and

$$\text{Var}(Y \mid X = x) = E(Y^2 \mid X = x) - [E(Y \mid X = x)]^2 \quad \ldots (7\cdot6\cdot5)$$

where

$$E(Y^2 \mid X = x) = \sum_y y^2 P(Y = y \mid X = x) = \sum_y y^2 P_{Y|X}(y \mid x)$$

$$= \frac{\sum_y y^2 P(x, y)}{P_X(x)} \quad \ldots (7\cdot6\cdot6)$$

For two continuous r.v.'s X and Y with the joint p.d.f. f(x,y), the conditional expectation of X , given Y=y is defined for all y such that $f_Y(y) > 0$ as

$$E(X \mid Y = y) = \int_{-\infty}^{\infty} x f_{X|Y}(x \mid y)\, dx$$

$$= \frac{\int_{-\infty}^{\infty} x f(x, y)\, dx}{f_Y(y)} \quad \ldots (7\cdot6\cdot7)$$

The conditional variance of X, given Y=y is defined as in (7.6.2), where

$$E(X^2 \mid Y = y) = \int_{-\infty}^{\infty} x^2 f_{X|Y}(x \mid y)\, dx$$

$$= \frac{\int_{-\infty}^{\infty} x^2 f(x, y)\, dx}{f_Y(y)} \quad \ldots (7\cdot6\cdot8)$$

Now , the conditional expectation of Y, given X=x is defined for all x such that $f_X(x) > 0$ as

$$E(Y \mid X = x) = \int_{-\infty}^{\infty} y \, . \, f_{Y|X}(y \mid x)\, dy = \frac{\int_{-\infty}^{\infty} y\, f(x, y)\, dy}{f_X(x)}$$

$$\ldots (7{\cdot}6{\cdot}9)$$

The conditional variance of Y, given X=x is defined as in (7.6.5), where

$$E(Y^2 \mid X = x) = \int_{-\infty}^{\infty} y^2\, f_{Y|X}(y \mid x)\, dy$$

$$= \frac{\int_{-\infty}^{\infty} y^2 f(x, y)\, dy}{f_X(x)}, \; f_X(x) > 0 \qquad \ldots (7{\cdot}6{\cdot}10)$$

Example 7.6.1.

Let X and Y be jointly distributed by p.m.f.

$$P(x, y) = \frac{x + y}{21} \quad , x = 1, 2, 3 \text{ and } y = 1, 2$$

$$0 \qquad \text{otherwise.}$$

Find (i) Var (X | Y =1) and (ii) Var (Y|X = 2).

Solution :-

i. The joint p.m.f. of X and Y is given by

Y=y / X=x	1	2	Total $P_X(x)$
1	2/21	3/21	5/21
2	3/21	4/21	7/21
3	4/21	5/21	9/21
Total $P_Y(y)$	9/21	12/21	1

The conditional p.m.f. of X given Y=1 is given by

$X = x$	1	2	3
$P_{X\|Y}(x \mid y = 1)$	$\frac{2}{9}$	$\frac{3}{9}$	$\frac{4}{9}$

The conditional expectation of X given Y=1 is given by

$$E(X \mid Y = 1) = \sum_x x\, P_{X|Y}(x \mid y = 1)$$

$$= 1\left(\frac{2}{9}\right) + 2\left(\frac{3}{9}\right) + 3\left(\frac{4}{9}\right) = 20/9.$$

The conditional expectation $E(X^2 \mid Y = 1)$ is given by

$$E(X^2 \mid Y = 1) = \Sigma\, x^2 P_{X|Y}(x \mid y = 1)$$

$$= 1\left(\frac{2}{9}\right) + 4\left(\frac{3}{9}\right) + 9\left(\frac{4}{9}\right) = 50/9.$$

Then conditional variance, Var (X | Y=1) is

$$\text{Var}(X \mid Y = 1) = E(X^2 \mid Y = 1) - (E(X \mid Y = 1)^2$$

$$= \frac{50}{9} - \left(\frac{20}{9}\right)^2 = 50/81.$$

ii. The conditional p.m.f. of Y given X=2 is given by

Y = y	1	2
$P_{Y\|X}(y \mid X = 2)$	$\frac{3}{7}$	$\frac{4}{7}$

The conditional expectations E (Y | X = 2) and $E(Y^2 \mid X = 2)$ are given by

$$E(Y \mid X = 2) = \sum_y y\, P_{Y|X}(y \mid x = 2)$$

$$= 1\left(\frac{3}{7}\right) + 2\left(\frac{4}{7}\right) = \frac{11}{7},$$

$$E(Y^2 \mid X = 2) = \sum_y y^2 P_{Y|X}(y \mid X = 2)$$

$$= 1\left(\frac{3}{7}\right) + 4\left(\frac{4}{7}\right) = 19/7$$

Then, the conditional variance, Var (Y | X=2) is

$$\text{Var}(Y \mid X = 2) = E(Y^2 \mid X = 2) - (E(Y \mid X = 2))^2$$

$$= \frac{19}{7} - \left(\frac{11}{7}\right)^2 = \frac{11}{49}$$

Example 7.6.2.

Let X and Y be two jointly distributed r.v.'s with the p.d.f.

$$f(x,y) = \begin{cases} 8xy & , 0 < x < y < 1 \\ 0 & \text{otherwise}. \end{cases}$$

Find (i) Var (X|Y = 0.5) and (ii) Var (Y | X = 0.7)

Solution :

The marginal p.d.f.'s of X and Y are, respectively

$$f_X(x) = \int_x^1 f(x,y)\,dy = \int_x^1 8xy\,dy = 4 \times y^2 \Big]_x^1$$

$$= 4x[1 - x^2] \quad , \quad 0 < x < 1$$

and

$$f_Y(y) = \int_0^y f(x,y)\,dx = \int_0^y 8xy\,dx = 4x^2 y \Big]_0^y$$

$$= 4y^3 \quad , \quad 0 < y < 1.$$

The conditional expectations E (X|Y) and E (X^2 | Y) are

E (X^2 | Y) are

$$E(X|Y) = \int_0^y x\, f_{X|Y}(x|y)\,dx$$

$$= \int_0^Y x \cdot \frac{8xy}{4y^3}\,dx = \frac{2}{y^2}\int_0^y x^2\,dx$$

$$= \frac{2y^3}{3y^2} = \frac{2y}{3} \quad 0 < y < 1$$

For $y = 0.5$, then

$$E(X \mid Y = 0.5) = \frac{2}{3}\left(\frac{1}{2}\right) = \frac{1}{3},$$

and

$$E(X^2 \mid Y) = \int_0^y x^2 \cdot \frac{2x}{y^2}\, dx = \frac{2}{4y^2}\, y^4 = \frac{y^2}{2}$$

$$E(X^2 \mid Y = 0.5) = \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{8}.$$

Therefore

$$Var(X \mid Y = 0.5) = E(X^2 \mid Y = 0.5) - (E(X \mid Y = 0.5))^2$$

$$= \frac{1}{8} - \frac{1}{9} = \frac{1}{72}.$$

ii. The conditional expectations $E(Y|X)$ and $E(Y^2 \mid X)$ are

$$E(Y \mid X) = \int_x^1 y\, f_{Y|X}(y \mid x)\, dy$$

$$= \int_x^1 y \cdot \frac{2y}{1 - x^2}\, dy = \frac{2}{1 - x^2} \cdot \frac{y^3}{3}\Bigg]_x^1$$

$$= \frac{2(1 - x^3)}{3(1 - x^2)}, \quad 0 < x < 1.$$

For $x = 0.7$, then

$$E(Y \mid X = 0.7) = \frac{2(1 - 0.343)}{3(1 - 0.49)} = \frac{1.314}{1.53} = 0.859$$

$$E(Y^2 \mid X) = \int_x^1 y^2 \cdot \frac{2y}{1-x^2}\, dy = \frac{(1-x^4)}{2(1-x^2)}\, dy$$

For $x = 0{\cdot}7$, then

$$E(Y^2 \mid X = 0{\cdot}7) = \frac{0{\cdot}7599}{1{\cdot}02} = 0{\cdot}745$$

Therefore ,

$$\begin{aligned} Var(Y \mid X = 0{\cdot}7) &= E(Y^2 \mid X = 0{\cdot}7) - (E(Y \mid X = 0{\cdot}7))^2 \\ &= 0{\cdot}745 - (0{\cdot}859)^2 \\ &= 0{\cdot}745 - 0{\cdot}738 = 0{\cdot}017 \end{aligned}$$

7.7. Problems

1. For the following joint p.m.f. of the r.v.'s X and Y , find (i) $P(X \leq 1, Y=2)$, (ii) $P(X \leq 1)$, (iii) $P(Y=3)$, (iv) $P(Y \leq 3)$, (v) $P(X < 3, Y \leq 4)$ (vi) $F(1,3)$, (vii) $F(0,5)$ and (viii) $F(2,2)$.

X \ Y	1	2	3	4	5	6
0	0	0	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{16}$	$\frac{3}{32}$
1	$\frac{1}{16}$	0	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{1}{4}$	0
2	$\frac{1}{32}$	0	$\frac{3}{64}$	$\frac{1}{64}$	0	$\frac{1}{32}$

2. Let X and Y be jointly distributed with p.m.f.

$$P(x, y) = \frac{1}{27}(x + 3y) \text{ for } x = -1, 0, 1 \text{ and } y = 1, 2 .$$

i. Find the marginal p.m.f.'s of X and Y.
ii. Find the conditional p.m.f. of X given Y=1.
iii. Find the conditional p.m.f. of Y given X=0.
iv. Find F (-1,2) and F (1,1).
v. Find E (X), E (Y) and E (X Y).

3. Let X and Y be jointly distributed with p.m.f.

$$P(x, y) = \frac{1}{30}(x + y) \text{ for } x = 0, 1, 2, 3 \text{ and } y = 0, 1, 2 .$$

Show that
i. the marginal distribution of X is given by

$$P_X(x) = (x + 1)/10 \text{ for } x = 0, 1, 2, 3$$

ii. the marginal distribution of Y is given by

$$P_Y(y) = (2y + 3)/15 \text{ for } y = 0, 1, 2 .$$

Also find

iii. $P(X \mid Y = 1)$ and $P(Y \mid X = 3)$.
iv. $E(X + Y)$ and $E(XY)$.

4. Let X and Y be jointly distributed with p.d.f.

$$f(x, y = \frac{2}{3}(x + 2y) \text{ for } 0 < x < 1, 0 < y < 1$$

$$0 \quad \text{otherwise}.$$

Find
i. the marginal density functions of X and Y.
ii. the conditional probability function of X given Y.

iii. the conditional probability function of Y given X.

iv. F (x,y) and F (0.2, 0.7).

v. Are X and Y independent r.v.'s. ?

5. The two r.v.'s X and Y have the joint distribution function

$$F(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y}) & \text{for } x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

i. Find the probability that both X and Y will take on values less than 2.

ii. Find the probability that X will take on value less than 3 and Y will take on value less than 4.

iii. Find the marginal p.d.f.'s of X and Y.

iv. Are X and Y independent r.v.'s ?

6. Referring to problem 1, find Var (X) ; Var (Y), Cov (X,Y), E (X|Y =5) and Var (Y | X=0).

7. Referring to problem 2, find E(Y|X=0) , E(X|Y=2) and V(X|Y =2)

8. Referring to problem 4, find Cov (X,Y) , $V(X \mid Y)$ and $V(Y \mid X = 0{\cdot}3)$.

9. Verify Equation (7.4.9).

10. Let X and Y be two independent r.v.'s having the same geometric distribution with parameter p. Show that the conditional distribution of X given that X+Y=n is uniform.

11. Let X and Y be independent random variables which have the Poisson distribution. with parameter m_1 and m_2 , respectively. Also let Z=X+Y. Show that the conditional p.m.f. of X, given Z=z , is binomial with parameters n=z and $p = m_1 / (m_1 + m_2)$ for z=0, 1,2,..

12. Let X and Y be independent binomially distributed r.v.'s with parameters (n,p) and (m,p) respectively. Also let Z= X+ Y, find E(X|Z = z) for z = 0,1,2, ..., n+m.