Biostatistics – Spring 2026
Lecture 11: Comparison of Survival Curves Using the Log-rank Test

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Introduction

In the previous lectures, we learned how to estimate survival probabilities using life tables and the Kaplan–Meier estimator. After estimating survival in each group, an important practical question appears:

> Do the two survival curves differ significantly, or is the observed difference only due to chance?

To answer this question, we use the **log-rank test**. It is one of the most common methods for comparing survival experience between two groups.

By the end of this lecture, you should be able to:

- state the null and alternative hypotheses for comparing two survival curves,
- identify the observed and expected number of events at each event time,
- compute the log-rank test statistic conceptually,
- make a statistical decision,
- interpret the result in biomedical language.

1. Why Do We Need the Log-rank Test?

Suppose we have two treatment groups and we draw two Kaplan–Meier curves.

Sometimes one curve appears higher than the other. However, a visual difference alone is not enough.

We need a formal statistical test to decide whether the survival experience of the two groups is significantly different.

The log-rank test compares the **observed** number of events in each group with the **expected** number of events under the assumption that the two groups have the same survival distribution.

2. Hypotheses of the Log-rank Test

Let $S_1(t)$ and $S_2(t)$ denote the survival functions of the two groups.

Null hypothesis

$$H_0 : S_1(t) = S_2(t) \quad \text{for all } t$$

This means that the two groups have the same survival pattern over time.

Alternative hypothesis

$$H_A : S_1(t) \neq S_2(t) \quad \text{for at least one } t$$

This means that the survival curves differ at some point in time.

3. Basic Quantities Used in the Test

At each event time t_j, define:

- n_{1j}: number at risk in group 1 just before time t_j,
- n_{2j}: number at risk in group 2 just before time t_j,
- d_{1j}: number of events in group 1 at time t_j,
- d_{2j}: number of events in group 2 at time t_j.

Also define:

$$n_j = n_{1j} + n_{2j}$$

$$d_j = d_{1j} + d_{2j}$$

These are the total number at risk and the total number of events at event time t_j.

4. Expected Number of Events

Under the null hypothesis, the expected number of events in each group at time t_j is proportional to the number at risk in that group.

Thus:

$$E_{1j} = \frac{n_{1j}}{n_j} d_j$$

$$E_{2j} = \frac{n_{2j}}{n_j} d_j$$

Interpretation: If the two groups truly have the same survival experience, then the total events at time t_j should be divided between the two groups according to how many individuals are at risk in each group.

5. Variance and Test Statistic

The variance contribution at time t_j is:

$$v_j = \frac{n_{1j} n_{2j} d_j (n_j - d_j)}{n_j^2 (n_j - 1)}$$

Then the log-rank test statistic is:

$$Z = \frac{\sum_{j=1}^{r} (d_{1j} - E_{1j})}{\sqrt{\sum_{j=1}^{r} v_j}}$$

Under the null hypothesis, Z approximately follows the standard normal distribution. We can also use:

$$W = Z^2$$

which approximately follows a Chi-square distribution with 1 degree of freedom:

$$W \sim \chi^2_{(1)}.$$

6. Worked Example

Suppose the following event-time data are obtained for two groups:

t_j (months)	d_{1j}	n_{1j}	d_{2j}	n_{2j}
3	1	10	0	10
5	1	9	1	10
8	2	8	1	9
12	1	6	2	8

Step 1: Compute totals

$$d_j = d_{1j} + d_{2j}, \qquad n_j = n_{1j} + n_{2j}$$

t_j	d_{1j}	n_{1j}	d_{2j}	n_{2j}	d_j	n_j	E_{1j}
3	1	10	0	10	1	20	0.500
5	1	9	1	10	2	19	0.947
8	2	8	1	9	3	17	1.412
12	1	6	2	8	3	14	1.286

Step 2: Compute the numerator

$$\sum (d_{1j} - E_{1j}) = (1 - 0.500) + (1 - 0.947) + (2 - 1.412) + (1 - 1.286)$$

$$= 0.500 + 0.053 + 0.588 - 0.286 = 0.855$$

Step 3: Compute the variance terms

$$v_j = \frac{n_{1j}n_{2j}d_j(n_j - d_j)}{n_j^2(n_j - 1)}$$

The values are approximately:

$$v_1 = 0.250, \qquad v_2 = 0.494, \qquad v_3 = 0.745, \qquad v_4 = 0.835$$

So:

$$\sum v_j = 0.250 + 0.494 + 0.745 + 0.835 = 2.324$$

Step 4: Compute Z

$$Z = \frac{0.855}{\sqrt{2.324}} = 0.561$$

Then:

$$W = Z^2 = (0.561)^2 = 0.315$$

Step 5: Decision

At $\alpha = 0.05$, the two-sided critical value from the standard normal distribution is:

$$1.96$$

Since:

$$|0.561| < 1.96$$

we do not reject H_0.
Equivalently, using Chi-square with 1 degree of freedom:

$$0.315 < 3.841$$

so again we do not reject H_0.

Final interpretation

> There is not enough statistical evidence at the 5% level to conclude that the two groups have different survival curves.

7. Intuitive Interpretation of the Log-rank Test

The test does not directly compare only one time point. Instead, it compares the two groups over the **whole follow-up period**.
At each event time:

- we count the observed events,
- we calculate the expected events under equal survival,
- we combine the differences across all event times.

If the observed and expected values stay close, the groups are considered similar.
If they are consistently far apart, the survival curves are considered different.

8. When Is the Log-rank Test Appropriate?

The log-rank test is appropriate when:

- the outcome is time to event,
- survival data may include censoring,
- the goal is to compare two or more survival curves,
- the groups are independent.

It is commonly used in:

- clinical trials,
- cancer survival studies,
- treatment comparisons,
- follow-up studies.

9. Common Student Mistakes

- Thinking the log-rank test compares means only.
- Forgetting that the test uses all event times.
- Confusing observed events with expected events.
- Believing that visually different curves always mean significance.
- Forgetting that the null hypothesis is equality of survival functions.

10. Summary

- The log-rank test compares survival curves between groups.
- The null hypothesis states that the survival functions are equal.
- The test compares observed and expected numbers of events at each event time.
- The test statistic may be written as Z or as $W = Z^2$.
- A non-significant result means there is not enough evidence of a survival difference.

Homework (HW)

HW1

State the null and alternative hypotheses for comparing the survival curves of two treatment groups.

HW2

Explain in words what the expected number of events E_{1j} represents in the log-rank test.

HW3

If a study gives $Z = 2.30$ for the log-rank test at $\alpha = 0.05$, what is the decision?

HW4

Why is it not enough to compare two survival curves visually without using a test?