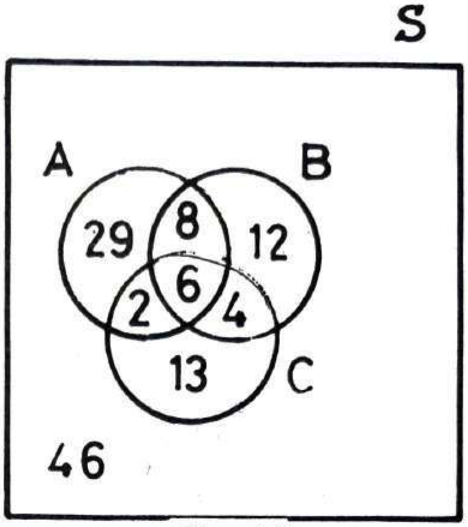

Fig. 4.3

i. $P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(AB)$

$$= \frac{45}{120} + \frac{30}{120} - \frac{14}{120} = \frac{61}{120}$$

ii. $P(B \text{ or } C) = P(B) + P(C) - P(BC)$

$$= \frac{30}{120} + \frac{25}{120} - \frac{10}{120} = \frac{45}{120}$$

iii. $P(A,B \text{ or } C) = P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

$$= \frac{45}{120} + \frac{30}{120} + \frac{25}{120} - \frac{14}{120} - \frac{10}{120} - \frac{8}{120} + \frac{6}{120} = \frac{74}{120}.$$

iv. $P(A \cup B \cup C)' = 1 - P(A \cup B \cup C)$

$$= 1 - \frac{74}{120} = \frac{46}{120}.$$

Proposition 4.2.3

For any n events $A_1, A_2 \ldots\ldots, A_n \in F$, we have

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j} P(A_i A_j) + \sum_{i<j<k}$$

$$P(A_i A_j A_k) - \ldots + (-1)^{n-1} P(A_1 A_2 \ldots A_n) \ldots \quad \ldots(4{\cdot}2{\cdot}4)$$

Proposition 4.2.3 can be proved by induction.

Example 4.2.4

In a 13-card bridge hand the probability that the hand will contain the A,K,Q,J of hearts is

$$P(\text{A,K,Q,J of hearts}) = \frac{C(48,9)}{C(52,13)} = \frac{429}{4165}$$

since A,K,Q, and J of hearts must be chosen, and the other 9 cards must be selected out of 48.

To find the probability of obtaining A,K,Q and J of at least one suit. Thus, if we let.

A_1 is the event that A,K,Q, and J of hearts is obtained,
A_2 is the event that A,K,Q, and J of diamonds is obtained,
A_3 is the event that A,K,Q, and J of spades is obtained,
and A_4 is the event that A,K,Q, and J of clubs is obtained,

So the required probability is
$P(A_1 \cup A_2 \cup A_3 \cup A_4)$.

We have

$$P(A_1) = P(A_2) = P(A_3) = P(A_4) = \frac{429}{4165}$$

$$P(A_1 A_2) = \frac{C(44,5)}{C(52,13)} = \frac{11}{1286390}$$

$$P(A_1 \ A_2 A_3) = \frac{C(40,1)}{C(52,13)}$$

The intersection of four events is empty, hence by (4.2.4), we get

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 4P(A_1) - 6P(A_1 A_2) + 4P(A_1 A_2 A_3)$$

$$= \frac{4C(48,9) - 6C(44,5) + 4C(40,1)}{C(52,13)}.$$

Now, by using Lemma 1.3.1 and Equation (4.2.2) we can show that the probability of the occurrence of at least one of a finte or countably infinte collection of events is less than or equal to the sum of the probabilities, i.e.

$$P(A_1 \cup A_2 \cup \ldots\ldots) \leq \sum_{i=1}^{\infty} P(A_i) \ldots\ldots\ldots\ldots \ (4.2.5)$$

Proposition 4.2.4

If $\{A_i, i \geq 1\}$ is either an increasing or decreasing (monotone) sequence of events, then

$$\lim_{i \to \infty} P(A_i) = P(\lim_{i \to \infty} A_i) \quad \ldots\ldots\ldots\ldots \ (4.2.6)$$

Proof:

If $A_1 \subset A_2 \subseteq A_3 \subseteq \ldots\ldots$, then their limit

$$\mathbf{A} = \bigcup_{i=1}^{\infty} A_i = \lim_{i \to \infty} A_i$$

$= A_1 \cup (A_2 / A_1) \cup (A_3 / A_2) \cup$ .... which are unions of disjoinmt events

Then, by Axiom (3), we have

$$P(A) = P(A_1) + \sum_{i=1}^{\infty} P(A_{i+1} / A_i)$$

$$= P(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n-1} P(A_{i+1} / A_i)$$

$$= P(A_1) + \lim_{n \to \infty} \sum_{i=1}^{n-1} [P(A_{i+1}) - P(A_i)] \text{, because}$$

$A_i \subseteq A_{i+1}$ for all i and $A_i A_{i+1} = A_i$

There fore $P(A) = \lim_{n \to \infty} P(A_n)$.

To prove the result for a decreasing sequence of events, if $A_1 \subseteq A_2 \subseteq A_3 \subseteq$ ...., then $A'_1 \subseteq A'_2 \subseteq A'_3 \subseteq$....... From the preceding equation, we have

$$P\left(\bigcup_{i=1}^{\infty} A'_i\right) = \lim_{n \to \infty} P(A'_n)$$

But $\bigcup_{i=1}^{\infty} A'_i = \left(\bigcap_{i=1}^{\infty} A_i\right)'$, we see that

$$P\left(\bigcap_{i=1}^{\infty} A_i\right)' = \lim_{n \to \infty} P(A'_n) = \lim_{n \to \infty} (1 - P(A_n))$$

$$1 - P\left(\bigcap_{i=1}^{\infty} A_i\right) = 1 - \lim_{n \to \infty} P(A_n)$$

or $$P\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} P(A_n)$$

where $\bigcap_{i=1}^{\infty} A_i$ is the limit of a decreasing sequence,

i.e $$\lim_{i\to\infty} A_i = \bigcap_{i=1}^{\infty} A_i.$$

So the result follows.

4.3 Conditional Probability & Baye's Theorem

We hve defined the probability of an event A relative to the sample space S, but many statements about chance take the form if B occurs, then the probability of A is "P", where A and B are two events. For instance, "the bus being on time" if "tomorrow is a sunny day"

Definition 4.3.1

Let (S,F,P) be probability space, and let the events $A,B \in F$ with $P(B) \neq 0$. The *conditional* probability of A given B, denoted by $P(A|B)$, is defined by

$$P(A\mid B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{P(AB)}{P(B)}, \quad P(B) \neq 0 \quad \cdots (4{\cdot}3/1)$$

If $P(B) > 0$, then $P(A|B)$ is uniquely defined by Equation (4.3.1). Otherwise, if $P(B) = 0$, then $P(A|B)$ can be taken to be any number in $[0,1]$.

If $P(B) > 0$, the conditional probability satisfies Kolmogo-rov's Axioms, i.e.

Axiom 1.

$$0 \leq P(A|B) \leq 1 \text{ for any } A \in F$$

Axiom 2.

$$p(S//B) = 1$$

Axiom 3.

Let $\{A_i, i=1,2 \ldots\}$ be a collection of mutually exclusive events, then

$$P\left(\bigcup_{i=1}^{\infty} A_i \mid B \right) = \sum_{i=1}^{\infty} P(A_i \mid B)$$

From Equation (4.3.1), we have

$$P(AB) = P(A|B).\ P(B) \text{ if } P(B) > 0 \qquad \ldots\ldots. (4.3.2)$$

In the same way, we have

$$P(AB) = P(B|A).\ P(A) \text{ if } P(A) > 0 \qquad \ldots\ldots..(4.3.3)$$

Equations (4.3.2) and (4.3.3) are called multiplication rules for two events A and B.

The multiplication rule can be extended by induction to obain a multiplication rule for any finite number of events. That is

$$P(A_1 A_2 \ldots\ldots A_n) = P(A_1) . P(A_2|A_1).P(A_3|A_1A_2) \ldots. P(A_n|A_1 \ldots\ldots A_{n-1}) \qquad \ldots\ldots (4.3.4)$$

Example 4.3.1

A fair coin is tossed three times. Find the probability of getting 2 heads given that the first toss shows a head?

Solution

Here S = {HHH, HHT, HTH, THH, TTH, THT, HTT, TTT}

Let

A = the event of getting 2 heads,
and B = the first toss shows a head.

Then, the required probability is P(A|B) which is

$$P(A|B) = \frac{P(AB)}{P(B)}$$

A = {HHT, THH, HTH};
B = {HHH, HHT, HTH, HTT};

and

$AB = \{HHT, HTH\}$

Therefore,

$$P(A|B) = \frac{218}{418} = \frac{1}{2}.$$

Note that we can solve this example if we are looking for reduced sample space S_1 which contains all the elements that satisfied the condition given, and hence find the required probability. That is

$S_1 = \{HHH, HHT, HTH, HTT\} = B$

$$P(A|B) = \frac{h}{n} = \frac{2}{4} = \frac{1}{2}.$$

Example 4.3.2

Two 4 's and two 7 's are arranged at random to form 4-digit numbers. We assume that all numbers are equally likely If the 4- digit number is odd., what is the probability that the two 4's are together.

Solution

Here S contains $\frac{4!}{2!2!}=6$ elements, that is

$S=\{4477, 4774, 7744, 7447, 7474, 4747\}$.

Then

$A=\{4477, 7744, 7447\}$,

$B=\{4477, 7447, 4747\}$,

and $AB=\{4477, 7447\}$

The required probability is

$$P(AIB)= \frac{P(AB)}{P(B)}$$

$$\frac{2/6}{3/6} = \frac{2}{3}.$$

Let A and B be two events, P(A) can be expressed in terms of conditiond probabilities $P(A|B)$ and $P(A|B')$ as

$P(A) = P(A|B) . P(B) + P(A|B') . P(B')$. (4. 3. 5)

Because $A = A \cap S$

$= A \cap (B \cup B')$

$= (A \cap B) \cup (A \cap B')$ which is a union of mutually exclusive evenl

$\therefore P(A) = P(AB) + P(AB')$

$= P(A \mid B) . P(B) + P(A \mid B') . P(B')$

Thus P(A) is a weighted average of the conditional probabilities $P(A|B)$ and $P(A|B')$

Equation (4.3.5) is a special case of a total probability theorem, which is stated as

Theorem 4.3.1- Total Probability Theorem

Let $A_1, A_2, \ldots\ldots ; A_n$ be a partition of the sample space S, with $P(A_i) > 0$ for all i .For any event E, we have.

$$P(E) = \sum_{i=1}^{n} P(E \mid A_i) . P(A_i) \qquad \ldots (4\cdot3\cdot6)$$

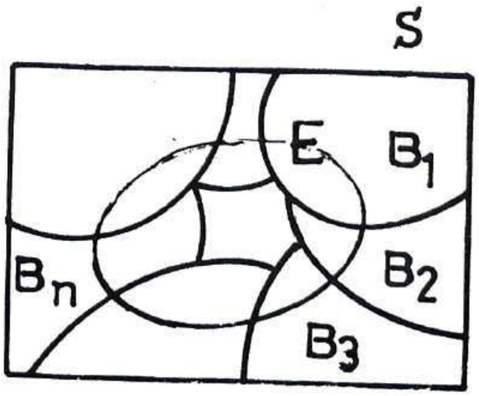

Fig 4.4

Proof :

Since $\bigcup_{i=1}^{n} A_i = S$, then

$$E = E \cap S = E \cap \left(\bigcup_{i=1}^{n} A_i \right)$$

$$= \bigcup_{i=1}^{n} (E \cap A_i)$$

Since A_1, A_2, ..., A_n are mutually exclusive events, then $A_1 \cap E$, $A_2 \cap E$, ... , and $A_n \cap E$ are also mutually exclusive events.

By Proposition 4.2.2, we have

$$P(E) = \sum_{i=1}^{n} P(EA_i)$$

$$= \sum_{i=1}^{n} P(E \mid A_i) . P(A_i) .$$

Example 4.3.3.

Box I contains 5 white balls and 7 red balls and box II contains 6 white and 4 red balls. A box is chosen at random and then a ball is drawn from it. Find the probability that the ball is red.

Solution

The boxes are equally likely to be chosen, then

$$P(I) = P(II) = \frac{1}{2} .$$

The required probability, P(R) can be obtained from equation (4.3.5),