8

Chapter Eight

Generating Functions

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8.1 Probability Generating Function - Mean and Variance

Definition 8.1.

Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. Using a variable t , we define the function

$$A(t) = a_0 + a_1 t + a_2 t^2 + \ldots = \sum_{k=0}^{\infty} a_k t^k \qquad \ldots (8\cdot1\cdot1)$$

If this power series converges in some interval $-t_0 < t < t_o$ then A (t) is called the *generating function* of the sequence $\{a_k\}$, k=0,1,2, ...

If we differentiate (8.1.1) k times with respect to t, and putting t = 0 and dividing by k ! , we get a_k , i.e.

$$a_k = \frac{1}{k!} \left. \frac{d^k A(t)}{dt^k} \right|_{t=0} \qquad \ldots (8\cdot1\cdot2)$$

For example , if $a_k = 1$ for all k , then

$$A(t) = 1 + t + t^2 + \ldots = \sum_{k=0}^{\infty} t^k = \frac{1}{1-t}, |t| < 1 .$$

If $a_k = \frac{1}{k!}$, k = 0, 1, 2, ..., then

$$A(t) = \sum_{k=0}^{\infty} \frac{t^k}{k!} = e^t.$$

Definition 8.1.2.

Suppose that X is a random variable which assumes non-negative integral - valued 0,1,2, ... and let $P_k=P\{X=k\}$, k=0,1, ... with $\sum_{k=0}^{\infty} P_k = 1$. If we let $a_k = P_k$ in (8·1·1)

then the probability generating function of the r.v. X (or simply p.g.f.) is defined as

$$P(t) = \sum_{k=0}^{\infty} P_k t^k = \sum_{k=0}^{\infty} P(X = k)\, t^k = E(t^X) \qquad \text{... (8·1·3)}$$

where $E(t^X)$ is the expectation of the function t^X of the r.v. X.

The series $\sum_{k=0}^{\infty} P_k t^k$ converges for at least $|t| < 1$.

Clearly $P(0) = P_0$ and $P(1) \sum_{k=0}^{\infty} P_k = 1$.

Now, if we differentiate (8.1.2.) with respect to t , and put t=1 , we get

$$P'(t) = \sum_{k=1}^{\infty} k\, P_k t^{k-1} \qquad \text{... (8·1·4)}$$

and

$$P'(1) = \sum_{k=1}^{\infty} k\, P_k = E(X).$$

Therefore, the mean of the r.v. X is equal to $P'(1)$.
Again differentiate (8.1.4) with respect to t, and put t=1, we get

$$P''(t) = \sum_{k=2}^{\infty} k(k-1) P_k t^{k-2}$$

and

$$P''(1) = \sum_{k=2}^{\infty} k(k-1) P_k = E(X(X-1)).$$

Thus, $P''(1) = EX^2 - EX$, or

$$EX^2 = P''(1) + P'(1)$$

Hence, the variance of X is given by

$$\text{Var}(X) = EX^2 - (EX)^2 = P''(1) + P'(1) - [P'(1)]^2$$

Example 8.1.1.

Let X be a r.v. with a geometric distribution

$$P_k = pq^k,\ k = 0, 1, 2, \ldots;\ p + q = 1.$$

Then, the p. g. f. of X is

$$P(t) = \sum_{k=0}^{\infty} P_k t^k = \sum_{k=0}^{\infty} pq^k t^k = p \sum_{k=0}^{\infty} (qt)^k$$

$$= \frac{p}{1 - qt}.$$

Hence

$$P'(t) = \frac{pq}{(1-qt)^2} \text{ and } P''(t) = \frac{2pq^2}{(1-qt)^2}.$$

Therefore ,

$$E(X) = P'(1) = \frac{q}{p},$$

$$P''(1) = \frac{2q^2}{p^2}, \text{ then}$$

$$\text{Var}(X) = P''(1) + P'(1) - [P'(1)]^2$$

$$= \frac{2q^2}{p^2} + \frac{q}{p} - \left(\frac{q}{p}\right)^2$$

$$= q / p^2 .$$

Example 8.1.2.

Let X have a binomial distribution with parameters n and p, then the p.g.f. of X is

$$P(t) = \sum_{k=0}^{n} C(n, k)\, p^k q^{n-k} t^k$$

$$= \sum_{k=0}^{n} C(n, k)(pt)^k q^{n-k}$$

$$= (pt + q)^n, \text{ by binomial theorem .}$$

Hence

$$P'(t) = np(pt + q)^{n-1} \text{ and } P''(t) = n(n-1)p^2(pt + q)^{n-2}$$

Thus

$$EX = P'(1) = np,$$

and

$$P''(1) = n(n-1)p^2$$

Therefore

$$\begin{aligned}\text{Var}(X) &= P''(1) + P'(1) - [P'(1)]^2 \\ &= n(n-1)p^2 + np - n^2p^2 \\ &= npq.\end{aligned}$$

Remarks :

1. The probabilities $P_k, k \geq 0$ can be uniquely determined from P(t) as follows

$$P_k = \frac{1}{k!} \left. \frac{d^k P(t)}{dt^k} \right|_{t=0}$$

2. The k- th factorial moments of X, E [X (X-1) ... (X-k+1)] can be found as

$$E[X(X-1)\ldots(X-k+1)] = \left. \frac{d^k P(t)}{dt^k} \right|_{t=1}, \text{ for}$$

$k = 1, 2, 3, \ldots$

Theorem 8.1.1.

Let Y = mX+n, where m and n are positive integers, with $m \neq 0$, the p.g.f. of Y, $P_Y(t)$ is given by

$$P_Y(t) = t^n P_X(t^m) \qquad \ldots (8{\cdot}1{\cdot}5)$$

where $P_X(t)$ is the p. g. f. of X .

Proof. We have

$$\begin{aligned}P_Y(t) &= E(t^Y) = E(t^{mX+n}) \\ &= E(t^n . t^{mX}) = t^n E(t^{mX}) \\ &= t^n E((t^m)^X) = t^n P_X(t^m)\end{aligned}$$

For example, if X has a binomial distribution with parameters n and p , then the p.g.f. of $Y = 3X + 4$ is given by

$$P_Y(t) = t^4 P_X(t^3)$$
$$= t^4 (pt^3 + q)^n.$$

Theorem 8.1.2.

Let X and Y be two independent r.v.'s with p.g.f.'s $P_X(t)$ and $P_Y(t)$, respectively, Let Z=X+Y, then the p.g.f. of Z is

$$P_Z(t) = P_X(t) . P_Y(t) \quad \ldots (8 \cdot 1 \cdot 6)$$

Proof

We have

$$P_Z(t) = E(t^Z) = E(t^{X+Y})$$
$$= E(t^X . t^Y) = E(t^X) . E(t^Y)$$ since X and Y are independent r.v.'s, then t^X and t^Y are also independent. Therefore,

$$P_Z(t) = P_X(t) P_Y(t).$$

Example 8.1.3.

Let X_1 and X_2 be two independent Poisson variates with parameters m_1 and m_2 , respectively , then

$$P_{X_1}(t) = \sum_{k=0}^{\infty} e^{-m_1} \frac{m_1^k t^k}{k!} = e^{-m_1} \sum_{k=0}^{\infty} \frac{(m_1 t)^k}{k!}$$

$$= e^{-m_1} . e^{m_1 t} = e^{m_1 (t-1)}$$

In the same way,

$$P_{X_2}(t) = e^{m_2(t-1)}$$

If we let $Z = X_1 + X_2$ then the p.g.f. of Z is

$$P_Z(t) = P_{X_1}(t) \cdot P_{X_2}(t) = e^{m_1(t-1)} \cdot e^{m_2(t-1)}$$
$$= e^{(m_1+m_2)(t-1)}.$$

Remark :

Theorem 8.1.2. can be extended to the sum of n independent r.v.'s $X_1, X_2, ..., X_n$, , that is , if $Z = X_1 + ... + X_n$, then

$$P_Z(t) = P_{X_1}(t) P_{X_2}(t) ... P_{X_n}(t)$$
$$= \prod_{i=1}^{n} P_{X_i}(t)$$

If $X_1, X_2, ... X_n$ are identically independent distributed random variables with the commmmon p.g.f. P(t) , then $Z = X_1 + ... + X_n$ has a p.g.f.

$$P_Z(t) = [P(t)]^n.$$

8.2. Moment Generating Function

Definition 8.2.1.

The moment generating function (or simply, m.g.f.) of the r.v. X, is denoted by $M_X(t)$, and if it exists, is defined as

$$M_X(t) = E(e^{tX}) \qquad \text{... (8·2·1)}$$

$$= \sum_x e^{tx} \cdot P(x), \text{ when X is discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx, \text{ when X is continuous ,}$$

Where t is a real parameter. We assume that the right hand side of (8.2.1) is absolutely convegent. Thus

$$M_X(t) = E(e^{tX})$$

$$= E\left[1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \dots\right]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots$$

$$= 1 + tm_1 + \frac{t^2}{2!}m_2 + \frac{t^3}{3!}m_3 + \dots$$

$$= \sum_{r=0}^{\infty} \frac{t^r m_r}{r!} \qquad \dots (8\cdot2\cdot2)$$

where $m_r = E(X^r)$ is the r- th moment about the origin of the r.v.X.

Now, if we differentiate (8.2.2) with respect to t, k times and put t=0, we get

$$\left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0} = m_k \text{ for } k = 1, 2, \dots$$

Note

It is clear that

$P(t) = E(t^X) = E(e^{X\ln t}) = M_X(\ln t)$