Proof :

Since $\bigcup_{i=1}^{n} A_i = S$, then

$$E = E \cap S = E \cap \left(\bigcup_{i=1}^{n} A_i \right)$$

$$= \bigcup_{i=1}^{n} (E \cap A_i)$$

Since A_1, A_2, ..., A_n are mutually exclusive events, then $A_1 \cap E$, $A_2 \cap E$, ... , and $A_n \cap E$ are also mutually exclusive events.

By Proposition 4.2.2, we have

$$P(E) = \sum_{i=1}^{n} P(EA_i)$$

$$= \sum_{i=1}^{n} P(E \mid A_i) . P(A_i) .$$

Example 4.3.3.

Box I contains 5 white balls and 7 red balls and box II contains 6 white and 4 red balls. A box is chosen at random and then a ball is drawn from it. Find the probability that the ball is red.

Solution

The boxes are equally likely to be chosen, then

$$P(I) = P(II) = \frac{1}{2} .$$

The required probability, P(R) can be obtained from equation (4.3.5),

$$P(R) = P(R|I).P(I) + P(R|II).P(II)$$

$$= \frac{7}{12} \cdot \frac{1}{2} + \frac{4}{10} \cdot \frac{1}{2} = \frac{59}{120} .$$

Example 4.3.4

In a box there are 100 resistors having resistance and tolerance as shown in the table below. Let a resistor be selected at random from the box and assume each resistor has the same likelihood of being chosen. Define three events: A as "draw a 47 ohms resistor", B as "draw a resistor with 5% tolerance" and C as "draw a 100 ohms resistor". Find the conditional probabilities P(A|B), P(A|C) and P(B|C).

Resistance in ohms \ Tolerace	5%	10%	Total
22	10	14	24
47	28	16	44
100	24	8	32
Total	62	38	100

Solution
we have

$$P(A|B) = \frac{P(AB)}{P(B)}$$

$$= \frac{P(\text{47 ohms and } 5\% \text{ tolerance})}{P(5\% \text{ tolerance})}$$

$$= \frac{28/100}{62/100} = \frac{28}{62} ;$$

$$P(A \mid C) = \frac{P(AC)}{P(C)} = \frac{P(\text{47 ohms and 100 ohms})}{P(\text{100 ohms})}$$

$$= 0$$

and

$$P(B \mid C) = \frac{P(BC)}{P(C)} = \frac{P(5\,\%_{\circ}\text{ tolerance and 100 ohms})}{P(\text{100 ohms})}$$

$$= \frac{24/100}{32/100} = \frac{24}{32}.$$

Theorem 4.3.2 Baye's Theorem

Under the assumptions of total probability theorem, we have

$$P(B_i \mid E) = \frac{P(B_i E)}{P(E)}, \; P(E) > 0, \text{ for } i = 1, 2, \ldots, n$$

One form of Bayes' Theorem is

$$P(B_i \mid E) = \frac{P(E \mid B_i) \cdot P(B_i)}{\sum_{i=1}^{n} P(E \mid B_i) \cdot P(B_i)}, \; i = 1, 2, \ldots, n \quad \ldots (4 \cdot 3 \cdot 6)$$

The theorem is named after Thomas Bayes (1702-1761), an English Philosopher.

The conditional probabilities $P(B_i \mid E)$ are called *posteriori* probabilities, since they apply after the performance of the experiment when some event E is obtained.

Example 4.3.5

A box contains three coins, one is fair, one is two headed, and one coin is weighted so that the probability of head appearing is $\frac{1}{3}$. A coin is selected at random and tossed once.

1. Given that the result is heads, find the probability that the two-headed coin was chosen.
2. Given that the result is tails, find the probability that the weighted coin was chosen.

Solution

The experiment is represented in Fig. 4.5 as a tree-diagram. Note that I refers to the fair coin, II to the two-headed coin and III to the weighted coin. Each coin has a probability $\frac{1}{3}$ to be chosen.

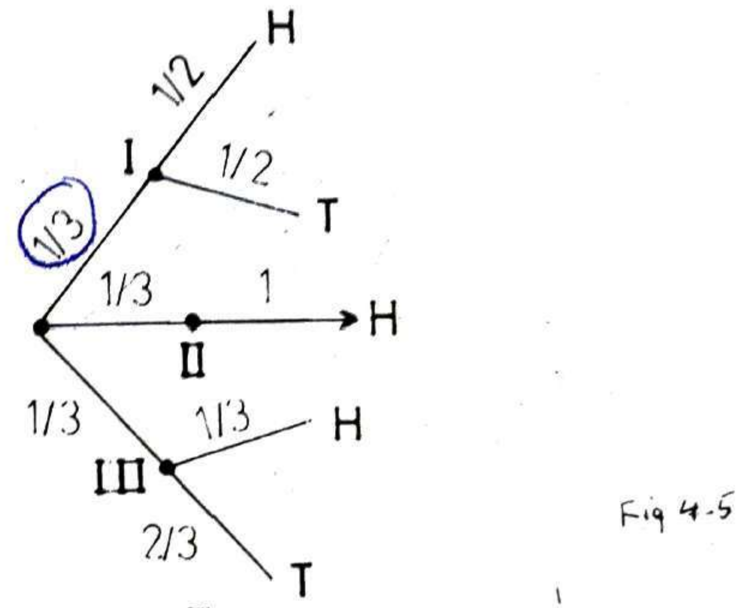

Fig. 4.5

1. $P(II \mid H) = \dfrac{P(H \mid II) \cdot P(II)}{P(H)}$

$$= \frac{1 \cdot \frac{1}{3}}{\frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3}} = \frac{\frac{1}{3}}{\frac{11}{18}}$$

$$= \frac{6}{11}.$$

2. $P(T) = 1 - P(H) = 1 - \frac{11}{18} = \frac{7}{18}$

$$P(III|T) = \frac{P(T|III).P(III)}{P(T)}$$

$$= \frac{\frac{2}{3} \cdot \frac{1}{3}}{\frac{7}{18}} = \frac{4}{7}.$$

Example 4.3.6

Two factories manufacture electric bulbs, 7% of the bulbs from factory A and 10%, from factory B are defective. Factory B produce three times as many bulbs as factory A each week. A bulbs is chosen at random from a week's production.

1. What is the probability that the bulbs is satisfactory?
2. If the bulbs is defective, what is the probability that it came from factory A?

Solution

Let D = the event that the chosen bulb is defective,
then D′ = the event that the chosen bulb is satisfactory.

we have, P(B) = 3P(A), then
P(A) + 3P(A) =1

$$P(A) = \frac{1}{4}.$$

1. By applying Equation (4.3.5), we get

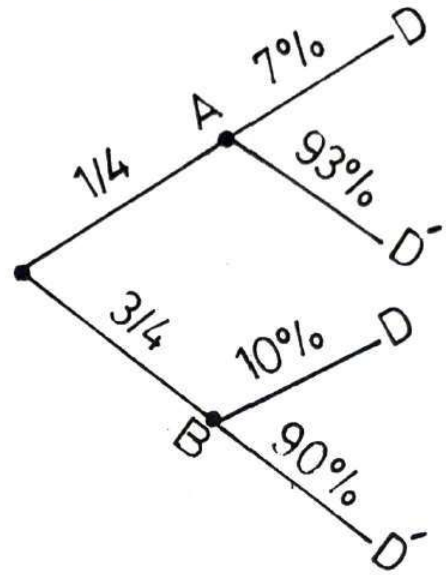

Fig. 4.6

$$P(D') = P(D'|A).P(A) + P(D'|B).P(B)$$

$$= \frac{93}{100} \cdot \frac{1}{4} + \frac{90}{100} \cdot \frac{3}{4} = \frac{363}{400} \cdot$$

$$2 \cdot \ P(A|D) = \frac{P(D|A) \cdot P(A)}{P(D)} = \frac{P(D|A) \cdot P(A)}{P(D|A) \cdot P(A) + P(D|B).P(B)}$$

we have $P(D) = 1 - P(D') = 1 - \frac{363}{400} = \frac{37}{400}$,

Therefore ,

$$P(A|D) = \frac{\frac{7}{100} \cdot \frac{1}{4}}{\frac{37}{400}} = \frac{7}{37} \cdot$$