Where t is a real parameter. We assume that the right hand side of (8.2.1) is absolutely convegent. Thus

$$M_X(t) = E(e^{tX})$$

$$= E\left[1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \dots\right]$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots$$

$$= 1 + tm_1 + \frac{t^2}{2!}m_2 + \frac{t^3}{3!}m_3 + \dots$$

$$= \sum_{r=0}^{\infty} \frac{t^r m_r}{r!} \qquad \dots (8{\cdot}2{\cdot}2)$$

where $m_r = E(X^r)$ is the r- th moment about the origin of the r.v.X.

Now, if we differentiate (8.2.2) with respect to t, k times and put t=0, we get

$$\left.\frac{d^k M_X(t)}{dt^k}\right|_{t=0} = m_k \text{ for } k = 1, 2, \dots$$

Note

It is clear that

$$P(t) = E(t^X) = E(e^{X\ln t}) = M_X(\ln t)$$

Now, if we let $Y = cX$, where c is a constant, then from (8.2.1) we have

$$M_Y(t) = E(e^{tY}) = E(e^{tcX}) = E[(e^{tc})^X]$$
$$= M_X(tc) \qquad \ldots (8{\cdot}2{\cdot}3)$$

Theorem 8.2.1.

For any constants a and b , the m.g.f. of $Y = \dfrac{X + a}{b}$ is given by

$$M_Y(t) = e^{\frac{a}{b}t} . M_X\left(\frac{t}{b}\right) \qquad \ldots (8{\cdot}2{\cdot}4)$$

where $M_X(t)$ is the m.g.f. of X.

Proof.

We have

$$M_Y(t) = E(e^{tY}) = E[e^{t(\frac{X+a}{b})}]$$

$$= E[e^{\frac{a}{b}t} . e^{\frac{t}{b}X}] = e^{\frac{a}{b}t} . E(e^{\frac{t}{b}X})$$

$$= e^{\frac{a}{b}t} . M_X\left(\frac{t}{b}\right).$$

Theorem 8.2.2.

Let X and Y be two independent r.v.'s with the m.g.f.'s $M_X(t)$ and $M_Y(t)$, respectively . Then , the m.g. f. of $Z = X + Y$ is given by

$$M_Z(t) = M_{X+Y}(t) = M_X(t) . M_Y(t) \qquad \ldots (8{\cdot}2{\cdot}5)$$

Proof :

We have

$$M_Z(t) = E(e^{tZ}) = E[e^{t(X+Y)}]$$
$$= E[e^{tX+tY}] = E[e^{tX} . e^{tY}]$$
$$= E(e^{tX}) E(e^{tY}), \text{ since X and Y are independent}$$

r.v.'s , then e^{tX} and e^{tY} are also independent r.v.'s. Hence, we get

$$M_Z(t) = M_X(t) . M_Y(t) .$$

Theorem 8.2.2 can be extended to the sum of n independent r.v.'s $X_i, X_2, \dots, X_n$, that is , if $Z = X_1 + \dots + X_n$, then the m.g.f. of Z is

$$M_Z(t) = \prod_{i=1}^{n} \cdot M_{X_i}(t),$$

where $M_{X_i}(t)$ is the m.g.f. of X_i , i = 1,2, ... n.

If $Z = \sum_{i=1}^{n} X_i$ is, the sum fo n idenlically independent r . v . 's X_{is}' with the common m . g . f . $M_X(t)$, then

$$M_Z(t) = [M_X(t)]^n$$

Example 8.2.1.

Let X be a r.v. having a Poisson distribution with parameter m, then the m.g.f. of X is

$$M_X(t) = \sum_{x=0}^{x} e^{tx} \frac{e^{-m}m^x}{x!} = e^{-m} \sum_{x=0}^{x} \frac{(me^t)^x}{x!}$$
$$= e^{-m} . e^{me^t} = e^{m(e^t - 1)}$$

To find the mean and the variance of X , we differentiate $M_X(t)$ twice with respect to t, and put t=0, thus we get

$$\left.\frac{dM_X(t)}{dt}\right|_{t=0} = e^{m(e^t-1)} . me^t \Big|_{t=0}$$

$$= m$$

and

$$\left.\frac{d^2M_X(t)}{dt^2}\right|_{t=0} = e^{m(e^t-1)} . m^2e^{2t} + me^t . e^{m(e^t-1)} \Big|_{t=0}$$

$$= m^2 + m .$$

Therefore ,

$E(X) = m , EX^2 = m^2 + m .$

Hence

$Var(X) = m .$

Example 8.2.2.

Let X have an exponential distribution

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & \text{otherwise} . \end{cases}$$

Then , the m.g.f. of X is

$$M_X(t) = \int_0^\infty e^{tx} . \lambda e^{-\lambda x} dx = \int_0^\infty \lambda e^{-x(\lambda - t)} dx$$

$$= \frac{-\lambda}{\lambda - t} e^{-x(\lambda - t)} \Big]_0^\infty$$

$$= \frac{\lambda}{\lambda - t}$$

If we differentiate $M_x(t)$ twice with respect to t, and put t=0, we get

$$\left.\frac{dM_x(t)}{dt}\right|_{t=0} = \left.\frac{\lambda}{(\lambda - t)^2}\right|_{t=0} = \frac{1}{\lambda}$$

and

$$\left.\frac{d^2 M_x(t)}{dt^2}\right|_{t=0} = \left.\frac{2\lambda}{(\lambda - t)^3}\right|_{t=0} = \frac{2}{\lambda^2}$$

Therefore,

$$E(X) = \frac{1}{\lambda}, \; EX^2 = \frac{2}{\lambda^2}.$$

thus $\mathrm{Var}(x) = \dfrac{1}{\lambda^2}$

Now, if we let $Z = X_1 + \ldots + X_n$ be the sum of n independent r.v.'s each having an exponential distribution with p.d.f.

$$f(x) = \lambda e^{-\lambda x}, x > 0,$$

then, the m.g.f. of Z is

$$M_Z(t) = \frac{\lambda}{\lambda - t} \cdot \frac{\lambda}{\lambda - t} \cdots \frac{\lambda}{\lambda - t} = \frac{\lambda^n}{(\lambda - t)^n}$$

If we differentiate $M_Z(t)$ twice with respect to t, and put t=0, we get

$$E(Z) = \frac{n}{\lambda} \text{ and } \mathrm{Var}(Z) = \frac{n}{\lambda^2}.$$

Example 8.2.3.

Let Z have a standard normal disstribution, i.e.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty$$

Then the m.g.f. of Z is

$$M_Z(t) = \int_{-\infty}^{\infty} e^{tz} \cdot \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z^2 - 2tz)} \, dz$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z - t)^2} \cdot e^{t^2/2} \, dz$$

$$= e^{t^2/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(z-t)^2} \, dz$$

let z-t = u , then dz = du

$$M_Z(t) = e^{t^2/2} \cdot \int_{-\infty}^{\infty} \frac{e^{-u^2/2}}{\sqrt{2\pi}} \, du = e^{t^2/2}$$

Now, if we let $X \sim N(\mu, \sigma^2)$ then

$Z = \dfrac{X - \mu}{\sigma}$ has N (0 , 1) . There fore

$$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t)$$

$\therefore e^{t^2/2} = e^{-\frac{\mu}{\sigma} t} M_X(t/\sigma)$ by Theorem 8.2.1.

$$\therefore M_X\left(\frac{t}{\sigma}\right) = e^{\frac{\mu}{\sigma} t} \cdot e^{t^2/2}$$

OR $M_X(t) = e^{t\mu + \frac{t^2}{2}\sigma^2}$

So that, if $X \sim N(\mu, \sigma^2)$, then the m.g.f. of X is

$$M_X(t) = e^{\mu t + t^2\sigma^2/2}$$

Remarks:

1. There is a one-to -one correspondence between the moment generating functions and distribution functions when the m.g.f. exists.

2. If the m.g.f. of the distribution of a r.v. approaches that of the distribution of another r.v., then the distribution of the first r.v. approaches that of the second r.v. under the same limiting conditions.

8.3. Characteristic Function.

For some distributions, if the series $\Sigma e^{tx} P(x)$ or $\int e^{tx} f(x)dx$ does not converge absolutely, then the m.g.f. does not exist. In this case, a more useful function than the m.g.f. is the characteristic function (or simply c.f.) which is defined as follows

Defintion 8.3.1

The characteristic function (c.f.) of a r.v. X is denoted by $\phi_X(t)$, and is defined as

$$\phi_X(t) = E(e^{itx}) \qquad \text{...... (8.3.1)}$$

$$= \sum_x e^{itx} P(x) \quad \text{if X is a discrete r.v.,}$$

$$= \int_{-\infty}^{\infty} e^{itx} f(x)\,dx \text{ if X is a continuous r.v.,}$$

where $i = \sqrt{-1}$, and t is any real number. Notice that

$$\phi_X(t) = M_X(it).$$

clearly,

i. $\phi_X(0) = E(e^{\circ}) = E(1) = 1$.

ii. $|\phi_X(t)| \leq 1$ because, if X is a continuous r.v., then

$$|\phi_X(t)| = \left| \int_{-\infty}^{\infty} e^{itx} f(x)\,dx \right| \leq \int_{-\infty}^{\infty} |e^{itx}|\, f(x)\,dx$$

But $e^{itx} = \cos tx + i \sin tx$, then

$$|e^{itx}| = \sqrt{\cos^2 tx + \sin^2 tx} = 1.$$

Therefore,

$$|\phi_X(t)| \leq \int_{-\infty}^{\infty} f(x)\,dx = 1$$

The c.f. $\phi_X(t)$ always exists, since $|\phi_X(t)| \leq 1$.

The characteristic function also generates the moments of the r.v. X in the following way: Since

$$\phi_X(t) = E(e^{itX})$$

$$= E\left[1 + itX + \frac{(it)^2}{2!} X^2 + \frac{(it)^3}{3!} X^3 + \ldots\right]$$

$$= 1 + it\,E(X) + \frac{(it)^2}{2!} EX^2 + \frac{(it)^3}{3!} EX^3 + \ldots$$

$$= \sum_{k=0}^{\infty} \frac{(it)^k}{k!} E(X^k) \qquad \ldots (8{\cdot}3{\cdot}2)$$

If $m_r = E(X^r)$ exists (for r=1,2, ...). then if we differentiate (8.3.2) r times with respect to t and put t=0, we get

$$\left. \frac{d^r \phi_X(t)}{dt^r} \right|_{t=0} = i^r E(X^r), r = 1, 2, \ldots$$

Therefore,

$$E(X^r) = \frac{1}{i^r} \left. \frac{d^r \phi_X(t)}{dt^r} \right|_{t=0}$$

Theorem 8.3.1

1. The characteristic function of $Y = cX$, where c is constant, is given by

$$\phi_Y(t) = \phi_X(ct) \qquad \ldots (8 \cdot 3 \cdot 3)$$

2. The c.f. of $Z = aX + b$, where a and b are constants, is given by

$$\phi_Z(t) = e^{itb} \phi_X(at) \qquad \ldots\ldots (8.3.4)$$

Proof:

1. $\phi_Y(t) = \phi_{cX}(t) = E[e^{itcX}]$

$$= E[e^{i(ct)X}] = \phi_X(ct).$$

2. $\phi_Z(t) = \phi_{aX+b}(t) = E[e^{it(aX+b)}]$

$$= E[e^{itb} \cdot e^{i(ta)X}]$$

$$= e^{itb} E[e^{i(ta)X}] = e^{itb} \phi_X(at).$$