Example 4.3.7

A man is equally likely to choose any one of three routes A, B and C from his house to the railway station, and his choice of route is not influenced by the weather. If the weather is dry, the probabilities of missing the train by routes A, B and C are $\frac{1}{20}$, $\frac{1}{10}$, $\frac{1}{5}$, respectively. He sets out on a dry day and misses the train. What is the probability that the route chosen was C?

Solution

Let E= the event that a man missing a train, then the required probability is P(C|E).

We have $P(A) = P(B) = P(C) = \frac{1}{3}$.

Therefore,

$$P(C|E) = \frac{P(E|C)\cdot P(C)}{P(E|A)\cdot P(A) + P(E|B)\cdot P(B) + P(E|C)\cdot P(C)}$$

$$= \frac{\frac{1}{5}\cdot\frac{1}{3}}{\frac{1}{20}\cdot\frac{1}{3} + \frac{1}{10}\cdot\frac{1}{3} + \frac{1}{5}\cdot\frac{1}{3}} = \frac{4}{7}.$$

Example 4.3.8

We are given two urns as follows:
Urn A contains 4 red and 3 white balls;
Urn B contains 3 red and 5 white balls.

An urn is selected at random; a ball is drawn and put into the other urn; then a ball is drawn from it. Find the probability that the drawn balls are of the same colour.

Solution :
Construct the following tree-diagram (Fig. 4.7)

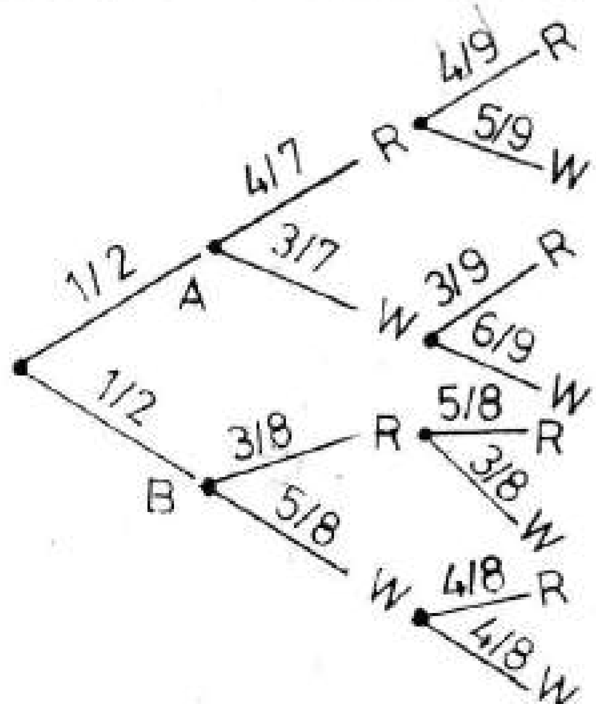

Fig. 4.7

P(two balls are of the same colour) =P(RR or WW)
= P(RR) +P(WW)

$$= \left(\frac{1}{2} \cdot \frac{4}{7} \cdot \frac{4}{9} + \frac{1}{2} \cdot \frac{3}{8} \cdot \frac{5}{8} \right) + \left(\frac{1}{2} \cdot \frac{3}{7} \cdot \frac{6}{9} \right.$$

$$\left. + \frac{1}{2} \cdot \frac{5}{8} \cdot \frac{4}{8} \right)$$

$$= \frac{4381}{8014} .$$

Example 4.3.9.

The chances that a doctor will diagnose a cancer disease correctly is 80%. The chance that a patient will die by his treatment after correct diagnosis is 25% and the chance of death by wrong diagnosis is 65%. One of this doctor's patients, who had a cancer died, what is the chance that his disease was diagnosed correctly?

Solution

Let E_1 = the event that the cancer is diagnosed correctly

E_2 = the event that a patient suffered from cancer dies.

Then, we have

$$P(E_1) = 0{\cdot}8,\ P(E_1') = 0{\cdot}2$$

$$\text{and } P(E_2 \mid E_1) = 0{\cdot}25,\ P(E_2 \mid E_1') = 0{\cdot}65$$

The required probability is $P(E_1 \mid E_2)$, which is

$$P(E_1 \mid E_2) = \frac{P(E_2 \mid E_1).\,P(E_1)}{P(E_2)}$$

$$= \frac{P(E_2 \mid E_1).\,P(E_1)}{P(E_2 \mid E_1)\,P(E_1) + P(E_2 \mid E_1').\,P(E_1')}$$

$$= \frac{(0{\cdot}25)(0{\cdot}8)}{(0{\cdot}25)(0{\cdot}8) + (0{\cdot}65)(0{\cdot}2)} = \frac{20}{33}$$

4.4 Independent Events

It was mentioned in Section 4.3 that the occurrence of some event B changes the probability that another event A occurs, and the original probability P(A) is replaced by P(A|B). If this probability remains unchanged, that is P(A|B) = P(A), then we call A and B *independent* This is well defined only if $P(B) > 0$

Since $P(AB) = P(A|B) . P(B)$, we see that A is independent of B if

$$P(AB) = P(A) . P(B) \quad \ldots\ldots\ldots (4.4.1)$$

To explain the idea of independent events, consider the following experiment. A ball is selected at random from a box containing 6 red and 4 white balls, its color is notice. After that a coin is tossed at random and its result observed. If A is the event that the ball is red, and B is the event that a head appeared, then A and B are obviously independent events.

Definition 4.4.1

Two events A and B are said to be (statistically) *independent* if Equation (4.4.1) holds.

Two events A and B that are not independent are said to be *dependent*.

Two events A and B are said to be *conditionally independent* with respect to E if

$$P(AB|E) = P(A|E), P(B|E) \quad \ldots\ldots\ldots (4.4.2)$$

Ramarks :

1. The conditional independence dose not imply independence.
2. If A and B are mutually exclusive events, then
   $P(AB) = 0.$ .... (4.4.3).

From Equations (4.4.1) and (4.4.3), we can easily establish that two events cannot be both mutually exclusive and (statistically) independent. Hence in order for two events to be independent their intersection must be nonempty.

For three events A , B and C to be independent, then we should have all the following conditions satisfied:

$$P(AB) = P(A).\ P(B),$$
$$P(AC) = P(A).\ P(C),$$
$$P(BC) = P(B).\ P(C).$$
and $P(ABC) = P(A).\ P(B)\ P(C)$.
which lead to the following definitions.

Definition 4.4.2

1. The events $A_1, A_2, \ldots, A_n$ are said to be independent if, for every subset $A_1, A_2, \ldots, A_r,\ r \le n$ of these events
$$P(A_1A_2 \ldots A_r) = P(A_1) \cdot P(A_2) \ldots P(A_r) \quad \ldots (4{\cdot}4{\cdot}4)$$

2. The events $A_1, A_2, \ldots$ are said to be independent if every finite subset of these events is independent.

3. If the events $A_1, A_2, \ldots$ have the property that
$$P(A_iA_j) = P(A_i) \cdot P(A_j) \text{ for all } i \neq j \quad \ldots (4{\cdot}4{\cdot}5)$$

then they are called *pairwise* independent. Note that the pairwise independent events are not necessarily independent as shown in the following example.

Example 4.4.1.

Let S = {123, 132, 321, 312, 231, 213, 111, 222, 333} and each of the nine elementary events in S occurs with probability $\frac{1}{9}$. Let A_k be the event that the k-th digit is 1, k = 1,2,3. Then

$$A_1 = \{111, 123, 132\}\ , \quad P(A_1) = \frac{3}{9} = \frac{1}{3},$$

$$A_2 = \{111, 213, 312\}\ , \quad P(A_2) = \frac{1}{3},$$

$$A_3 = \{111, 121, 211\} \quad , \quad P(A_3) = \frac{1}{3}$$

and

$$A_1A_2 = A_1A_3 = A_2A_3 = \{111\} , P(\{111\}) = \frac{1}{9}$$

We have $P(A_1A_2) = P(A_1) \cdot P(A_2) = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$,

$$P(A_1A_3) = P(A_1) \cdot P(A_3) = \frac{1}{9}$$

and $\quad P(A_2A_3) = P(A_2) \cdot P(A_3) = \frac{1}{9}$

So the events A_1 , A_2 and A_3 are *pairwise* ***independent*** but they are not independent because

$$A_1A_2A_3 = \{111\} ; P(A_1A_2A_3) = \frac{1}{9} \neq P(A_1) \cdot P(A_2) \cdot P(A_3)$$

$$= \frac{1}{3} \cdot \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{27}$$

Example 4.4.2.

A family has 2 children . Assume that **the birth** of a boy or a girl is equally **likely** to happen. **What** is the probability that the family **has** 2 boys?

Solution

The successive births are independent events, then

$$P(BB) = P(B) \cdot P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} .$$

Theorem 4.4.1.

If A and B are independent events, then A′ and B′ are also independent events.

Proof:

To show that $P(A'B') = P(A') \cdot P(B')$

We have

$$P(A'B') = P(A \cup B)' = 1 - P(A \cup B)$$

But $P(A \cup B) = P(A) + P(B) - P(AB)$.

Since A and B are two independent events, then $P(AB) = P(A)\ P(B)$, so we have

$$\begin{aligned} P(A'B') &= 1 - P(A) - P(B) + P(A)\ P(B) \\ &= (1 - P(A)) - P(B)\ (1 - P(A)) \\ &= (1 - P(A))\ (1 - P(B)) = P(A') \cdot P(B') \end{aligned}$$

Proposition 4.4.1.

If A and B are independent events, then

1. A and B′ are also independent events;
2. A′ and B are also independent events; and
3. $P(A \cup B) = 1 - P(A')\ P(B')$

Example 4.4.3.

The probability that A hits a target is $\frac{1}{4}$ and the probability that B hits a target is $\frac{1}{3}$.

1. If each fires once, what is the probability that both hit the target?

2. If each fires once, and the target is hit only once, what is the probability that A hits the target?

Solution

1. P(A hits the target) = $P(A) = \frac{1}{4}$, $P(A') = \frac{3}{4}$;

and P(B hits the target) = $P(B) = \frac{1}{3}$, $P(B') = \frac{2}{3}$

The two events are independent, then P(both hit the target)

$$= P(AB) = P(A) \cdot P(B) = \frac{1}{4} \cdot \frac{1}{3} = \frac{1}{12}$$

2. E = the event that the target is hit only once

$$P(E) = P(AB' \text{ or } A'B) = P(AB') + P(A'B)$$

$$= P(A) \cdot P(B') + P(A') \cdot P(B)$$

$$= \frac{1}{4} \cdot \frac{2}{3} + \frac{3}{4} \cdot \frac{1}{3} = \frac{5}{12}$$

and

$$P(A \mid E) = \frac{P(A \text{ and } E)}{P(E)} = \frac{P(AB')}{P(E)} = \frac{\frac{1}{4} \cdot \frac{2}{3}}{\frac{5}{12}} = \frac{2}{5}.$$

Example 4.4.4

Six married couples are involved in an aeroplane crash, and 5 people survive. What is the probability that the survivals include at least one married couple?

Solution

Let A_i= {ith couple survive} , i= 1,2, ... 6 The required probability is

$$P\left(\bigcup_{i=1}^{6} A_i\right) = \Sigma P(A_i) - \Sigma P(A_i A_j) + \Sigma P(A_i A_j A_k) \ldots$$

$$+ (-1)^6 P(A_1 A_2 \ldots A_6)$$

In this case, the intersection taken three or more at a time are empty, hence

$$P\left(\bigcup_{i=1}^{6} A_i\right) = \sum_{i=1}^{6} P(A_i) - \sum_{i \neq j} P(A_i A_j)$$

$$= 6 \cdot \frac{5}{12} \cdot \frac{4}{11} - 15 \cdot \frac{5}{12} \cdot \frac{4}{11} \cdot \frac{3}{10} \cdot \frac{2}{9}$$

$$= \frac{10}{11} - \frac{5}{33}$$

$$= \frac{25}{33}$$

Example 4.4.5.

A Problem in Statistics is given to the three students A, B and C whose chances of solving it are $\frac{2}{5}$, $\frac{3}{4}$ and $\frac{2}{3}$ respectively. What is the probability that the problem will be solved?

Solution

The problem will be solved if at least one student can solve it, so the required probability is

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

Since the three events A, B and C are independent, then

$$P(A \cup B \cup C) = \frac{2}{5} + \frac{3}{4} + \frac{2}{3} - \frac{2}{5} \cdot \frac{3}{4} - \frac{2}{5} \cdot \frac{2}{3} - \frac{3}{4} \cdot \frac{2}{3} + \frac{2}{5} \cdot \frac{3}{4} \cdot \frac{2}{3} = \frac{19}{20}$$

We can solve this example in the following way:

$$P(A \cup B \cup C) = 1 - P(A \cup B \cup C)'$$

$$= 1 - P(A'B'C')$$

$$= 1 - P(A').P(B').P(C')$$

$$= 1 - \frac{3}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} = \frac{19}{20}.$$

Example 4.4.6.

There are 10 people in a room. What is the probability that at least 2 of them have the same birthmonth? (Assume that all birthmonths are equally likely).

Solution:

Let A = no 2 people have the same birthmonth.. The required probability is 1- P(A).

$$P(A) = \frac{12}{12} \cdot \frac{11}{12} \cdot \frac{10}{12} \cdot \frac{9}{12} \cdot \frac{8}{12} \cdot \frac{7}{12} \cdot \frac{6}{12} \cdot \frac{5}{12} \cdot \frac{4}{12} \cdot \frac{3}{12}$$

$$= \frac{P(12,10)}{(12)^{10}} = \frac{P(11,9)}{(12)^9}$$

Hence

P(at least 2 people have the same birthmonth)

$$= 1 - \frac{P(11,9)}{(12)^9}$$

This answer can be written in the form

$$1 - \left(1 - \frac{1}{12}\right)\left(1 - \frac{2}{12}\right)\left(1 - \frac{3}{12}\right) \cdots \left(1 - \frac{9}{12}\right)$$

4.5. Problems

1. If S = {a,b,c,d,}. Which function defines a probability space on S?

   i. $P(a) = \frac{1}{8}$, $P(b) = P(c) = \frac{1}{4}$, and $P(d) = \frac{1}{3}$

   ii. $P(a) = \frac{1}{2}$, $P(b) = \frac{1}{3}$ $P(c) = \frac{1}{4}$ and $P(d) = \frac{1}{5}$

   iii. $P(a) = P(c) = \frac{1}{3}$, and $P(b) = P(d) = \frac{1}{6}$

   iv. $P(a) = -0.2$, $P(b) = 0.5$, $P(c) = 0.3$ and $P(d) = 0.4$.

2. Let P be a probability function on $S = \{a_1, a_2, a_3\}$. Find $P(a_1)$ If

   i. $P(a_2) = \frac{1}{3}$ and $P(a_3) = \frac{1}{4}$

   ii. $P(a_2) = 3\,P(a_1)$, $P(a_3) = \frac{1}{5}$.

   iii. $P(\{a_2, a_3\}) = 2\,P(a_1)$

   iv. $P(a_3) = 2P(a_2)$ and $P(a_2) = 3\,P(a_1)$

3. A box contains 4 white , 6 red , 5 blue and 3 orange balls. Three balls are drawn at random from the box. What is the probability that
   a. two of them are white,
   b. one is red and 2 are orange,
   c. none of them is blue,
   d. all of them are either white or blue.

4. Two cards are drawn one after the other without replacement from a well-shuffled deck of 52 cards. Find the probability that they are

a. of spades
b. one is heart and the other is diamond
c. 10 of spades and J of club.
d. red cards.,
e. numbered cards,
f. picture cards.

5. Two prizes are to be given to two students who are chosen at random from the student Ayad, Bashir, Khalid, David and Sami. (1) Find the elements of the sample space . (2) Find the probability of the events:

E_1: Ayad receives a prize,
E_2 : either Bashir or Sami receives a prize.
E_3 : Khalid does not receives a prize.

and E_4 : either Khalid or David receives a prize but not both.

6. Five couples are to be assigned to 2 different rooms. Find the probability that

a. one room must have 2 couples.
b. one room must have at least 4 persons.

7. Prove Equation (4.2.3)

8. For any n events $A_1, A_2, \ldots, A_n \in F$, Show that

$$P\left(\bigcup_{i=1}^{n} A_i \right) \leq \sum_{i=1}^{n} P(A_i)$$

9. Out of 150 students in a certain college, 35 students failed Mathematics, 30 failed Probability, 20 failed Algebra, 15 failed both Mathematics and Probability, 12 failed both Probability and Algebra, 8 failed both Mathematics and Algebra, and 6 failed in all subjects. A student is chosen at random. What is the probability that he failed

i. Mathematics or Probability
ii. Probability or Algebra
iii. in one of these subjects.
iv only Mathematics.

10. Show that Axioms (1′) to (3′) are satisfied by the conditional probability

11. Prove the following laws.

a. $P(A \mid A) = 1$, (b) $P(\phi \mid B) = o$
c. If $E \subset F$, then $P(E|A) < P(F|A)$ for any event A.
d. $P(A' 1B) = 1 - P(AIB)$
e. $P(AUB \mid C) = P(A|C) + P(B|C) - P(AB|C)$

10. If $B_1, B_2, \ldots, B_n$ form a partition of the sample space S, then, for any event $A \subset S$,

$$P(A) = \sum_{i=1}^{n} P(A \cap B)$$

11. In a sample space S, three events A, B and C have the probabilities $P(A) = P(B) = \frac{1}{3}$, $P(C) = \frac{1}{4}$, $P(AB) = \frac{1}{6}$,

$P(AC) = \frac{1}{8}$, $P(BC) = 0$. Find $P(A \cup B \cup C)$.

12. In answering a question on a multiple choice test, a student either knows the answer or he guesses. Let P be the probability that he knows the answer and 1- P. the probability that he guesses. A ssume that a student who guesses the answer will be correct with probability $\frac{1}{m}$,

where m is the number of multiple choice alternatives. What is the conditional probability that a student knew the answer to a question, given that he answered it correctly?

13. A couple has 2 children. What is the probability that both are girls if the eldest is a girl?

14. Three cards are randomly selected without replacement, from an ordinary deck of 52 playing cards. Compute the conditional probability that the third card selected is a spade, given that the second and first cards are spades.

15. Box A contains 9 cards numbered 1 through 9, and box B contains five cards numbered 1 through 5. A box is chosen at random and a card drawn from it.

   i. What is the probability that the cards shows an odd number?
   ii. If the card shows an odd number, what is the probability that it comes from box A?

16. If A , B , and C are events for some random experiment Show that the probability that exactly one of the events A, B or C occurs is
$P(A) + P(B) + P(C) - 2P(AB) - 2P(AC) - 2\,P(BC) + 3P(ABC)$.

17. In a certain village 20% of the population has disease D. A test is administered which has the property that if a person has D, the test will be positive 90% of the time, and if he does not have D, the test will still be positive 30% of the time. All those whose test is positive are given a drug which invariably cures the disease, but produces a characteristic rash 25% of the time. Given that a person picked at random has the rash, what is the probability that he actually has D to begin with?

(Hint : see Fig. 4.8)

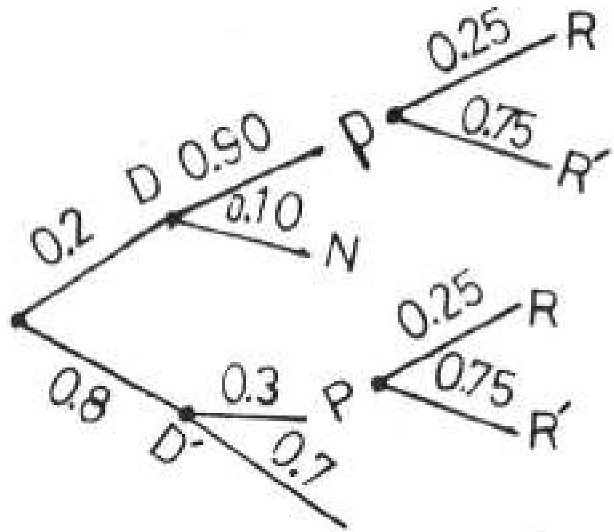

Fig. 4.8

18. In a certain town 45% of the people have black hair, 30% have brown eyes and 20% have both black hair and brown eyes. A person is selected at random from the town.

    i. If he has black hair, what is the probability that he also has brown eyes?
    ii. If he has brown eyes, what is the probability that he does not have black hair?

19. The probability that a man will live 30 more years is $\frac{1}{6}$, and the probability that his wife will live 30 more years is $\frac{1}{5}$. Find the probability that

    i. both will be alive in 30 years,
    ii. at least one will be alive in 30 years,
    iii. only the wife will be alive in 30 years.

20. Prove Proposition 4.4.1.

21. Box A contains 10 items of which 4 are defective and box B contains 8 items of which 3 are defective. An item is drawn at random from each box.

i. What is the probability that both items are good?

ii. What is the probability that one item is defective and one is not?

iii. If one is defective and one is not, what is the probability that the defective item came from box A?