Theorem- 8.3.2.

If X and Y are two independent random variables whose c.f s are $\phi_X(t)$ and $\phi_Y(t)$ respectively, then the c.f. of Z=X+Y is given by

$$\phi_Z(t) = \phi_{X+Y}(t) = \phi_X(t) \cdot \phi_Y(t) \qquad \text{......... (8.3.5)}$$

The proof of Theorem 8.3.2 is similar to that of Theorem 8.2.2.

Theorem 8·3·2 can be extended to n independent r. v.'s, that is, if $Z = X_1 + X_2 + \ldots + X_n$, then

$$\phi_Z(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) \ldots \phi_{X_n}(t). \qquad \text{... (8·3·6)}$$

If Z= $\sum_{i=1}^{n} X_i$ is the sum of n identically independent r.v.'s X_i's with the common c.f. $\phi_X(t)$, then

$$\phi_Z(t) = [\phi_x(t)]^n,$$

Example 8.3.1.

Let X be a r.v. with the p.m.f.

$$P(x) = \frac{1}{2^x} \quad , x=1,2,3, \ldots..$$

Then the c.f. of X is

$$\phi_X(t) = \sum_{x=1}^{x} e^{itX} \left(\frac{1}{2^x} \right) = \sum_{x=1}^{x} \left(\frac{e^{it}}{2} \right)^x$$

$$= \frac{\frac{e^{it}}{2}}{1 - \frac{e^{it}}{2}} = \frac{e^{it}}{2 - e^{it}}$$

Example 8.3.2.

Let X have a Poisson distribution with parameter m, then the c.f. of X is

$$\phi_X(t) = \sum_{x=0}^{\infty} e^{itx}. \frac{e^{-m} m^x}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{(me^{it})^x}{x!}$$

$$= e^{-m} e^{me^{it}} = e^{m(e^{it}-1)} .$$

To find the first and second moments of X we differentiate the c.f. twice with respect to t and put t=0, thus we get

$$\left.\frac{d\phi_X(t)}{dt}\right|_{t=0} = \left. im\, e^{it} . e^{m(e^{it}-1)} \right]_{t=0}$$

$$= im ,$$

Then, EX=m,

and

$$\left.\frac{d^2\phi_X(t)}{dt^2}\right|_{t=0} = \left[(im)^2 e^{2it} e^{m(e^{it}-1)} + i^2 me^{it} e^{m(e^{it}-1)} \right]_{t=0}$$

$$= i^2 (m^2 + m)$$

Then $EX^2 = m^2 + m$, and hence Var (X)=m.

Example 8.3.3.

Let X be uniformly distributed over the interval (a,b), then

$$f(x) = \begin{cases} 1/(b-a) & , a < x < b \\ 0 & \text{otherwise.} \end{cases}$$

Therfore, the c.f. of X is

$$\phi_X(t) = \int_a^b e^{itx} \frac{1}{b-a} dx$$

$$= \frac{e^{itx}}{it}\Big]_a^b = \frac{e^{itb} - e^{ita}}{it} .$$

Example 8.3.4.

If X has a Poisson distribution with parameter m. Find the c.f. of Y=3X+4, and the c.f. of Z=-2X+0.5.

Solution

From Example 8.3.2, we have

$$\phi_X(t) = e^{m(e^{it} - 1)} .$$

By Theorem 8.3.1 (2), we have

$$\phi_Y(t) = \phi_{3X+4}(t) = e^{4it} . \phi_X(3t)$$

$$= e^{4it}.e^{m(e^{3it} - 1)} ,$$

and

$$\phi_Z(t) = \phi_{-2X+0.5}(t) = e^{0.5it} . e^{m(e^{-2it} - 1)}$$

Example 8.3.5.

Let Z have a standard normal distribution, then. if we follow the same method as in Example 8.2.3, we can show that

$$\phi_Z(t) = \int_{-\infty}^{\infty} e^{itz} . \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

$$= e^{-t^2/2} .$$

Now, let $X \sim N(\mu, \sigma^2)$, then $Z = \frac{X - \mu}{\sigma}$ has a standard normal distribution. Therefore,

$$\phi_Z(t) = \phi_{\frac{X-\mu}{\sigma}}(t)$$

$$e^{-t^2/2} = e^{-it\frac{\mu}{\sigma}} . \phi_X\left(\frac{t}{\sigma}\right).$$

OR

$$\phi_X\left(\frac{t}{\sigma}\right) = e^{it\frac{\mu}{\sigma}} . e^{-t^2/2},$$

and hence

$$\phi_X(t) = e^{it\mu - \frac{t^2\sigma^2}{2}}$$

It follows that, if $X \sim N(\mu, \sigma^2)$ then the c.f. of X is $e^{it\mu - t^2\sigma^2/2}$.

8.4. Cumulant Generating Function

Definition 8.4.1.

The logarithm of characteristic function $K_X(t)$, is denoted by $\phi_X(t)$, and is defined as

$$K_X(t) = log_e \phi_X(t) = ln \phi_X(t).$$

From (8.3.2), we have

$$\phi_X(t) = 1 + \sum_{r=1}^{\infty} \frac{(it)^r}{r!} m_r$$

Putting $u = \sum_{r=1}^{\infty} \frac{(it)^r}{r!} m_r$, then

$$K_X(t) = ln(1 + u)$$

$$= u - \frac{u^2}{2} + \frac{u^2}{3} - \cdots\cdot$$

$$= \sum_{r=1}^{\infty} k_r \frac{(it)^r}{r!} \qquad \ldots (8{\cdot}4{\cdot}1).$$

The coefficients k_r in (8.4.1) are called cumulants or semi- invariants of the r.v. X. Comparing (8.3.2) and (8.4.1) we get.

$$\phi_X(t) = 1 + \sum_{r=1}^{\infty} \frac{(it)^r}{r!} m_r = \exp\left[\sum_{r=1}^{\infty} k_r \frac{(it)^2}{r!}\right]$$

$$= 1 + \sum_{r=1}^{\infty} \frac{(it)^r}{r!} k_r + \frac{1}{2!}\left[\Sigma \frac{(it)^r}{r!} k_r\right]^2 + \cdots\cdots$$

$$\ldots (8{\cdot}4{\cdot}2)$$

Comparing the coefficients of (8.4.2), we get

$$m_1 = k_1 ,$$
$$m_2 = k_2 + k_1^2 ,$$
$$m_3 = k_3 + 3 k_2 k_1 + k_1^3 ,$$
$$m_4 = k_4 + 4k_3 k_1 + 3k_2^2 + 6k_2 k_1^2 + k_1^4 . \qquad \dots(8{\cdot}4{\cdot}3)$$

From (8.4.3) we can get

$$k_1 = m_1 = E(X)$$

$$k_2 = m_2 - m_1^2 = \text{Var}(X) = \mu_2$$

$$k_3 = m_3 - 3m_2 m_1 + 2m_1^3 = \mu_3$$
$$k_4 = m_4 - 3m_2^2 - 4m_3 m_1 + 12m_1^2 m_2 - 6m_1^4$$

$$= \mu_4 - 3\mu_2^2 \qquad \dots(8{\cdot}4{\cdot}4)$$

From (8.4.3) and (8.4.4) we can conclude that if moments of nth order exist, all the cumulants of an order lower than n always exist.

Example 8.4.1

If X has a Poisson distribution with parameter m, then we have

$\phi_{X.}(t) = e^{m(e^{it}-1)}$ (see Example 8·3·2).

Thus

$$K_X(t) = \ln \phi_X(t) = \ln \{ e^{m(e^{it}-)} \}$$
$$= m(e^{it} - 1)$$

$$= m \left[1 + it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \dots - 1 \right]$$

$$= m \left[it + \frac{(it)^2}{2!} + \frac{(it)^3}{3!} + \dots \right].$$

Comparing the last equation by (8.4.1), we get

$k_r = m$ for all $r = 1, 2, \ldots$

Example 8.4.1

Let $X \sim N(\mu, \sigma^2)$, then

$$\phi_X(t) = e^{it\mu - \frac{t^2\sigma^2}{2}} \quad \text{(see Example 8·3·5)}.$$

Thus

$$K_X(t) = it\mu - \frac{t^2\sigma^2}{2} = it\mu + \frac{i^2t^2\sigma^2}{2}.$$

Comparing the last equation by (8·4·1) we get

$k_1 = \mu$, $k_2 = \sigma^2$ and $k_r = 0$ for $r > 2$.

8.5 Problems.

1. Let X be a r.v. with a p.m.f.

$$P(x) = Pq^{x-1}, x = 1, 2, 3, \ldots$$

Find the p.g.f. of (a) X, (b) X+1, and (c) 3X.

2. Let X_i , i=1,2, ..., n be mutually independent variables, each assuming the values 0,1,2, ..., a-1 with probabilities 1/a. Let $S_n = X_1 + X_2 + \ldots + X_n$ Show that the generating function of S_n is

$$P_{S_n}(t) = [1 - t^a)/a(1 - t)]^n.$$

3. Let X_i , i=1,2, ..., n be n independent Poisson variaties with parameters m_i , respectively. Let $S_n = X_1 + X_2 + \dots + X_n$, find the p.g.f. of S_n and calculate $Var(S_n)$.

4. If X is a r.v. with the p.g.f.

$$P(t) = \left(\frac{1}{4} + \frac{3}{4} t \right)^{10}$$

find (a) P(X=0), (b) P(X=4) and (c) Var (X).

5. If X is a r.v. with the p.g.f. P(t). Let Y=-X, show that the p.g.f. of Y is P(1/t).

6. Let X be a r.v. having a geometric distribution with parameter p. Find the m.g.f. of X, and hence calculate Var(X).

7. Let X be a r.v. having a binomial distribution with parameters n and p.

a. Find the m.g.f. of X, and hence calculate Var (X).

b. Find the m.g.f. of Y, where

$$Y = \frac{X - E(X)}{\sqrt{Var(X)}}.$$

8. Let X be a r.v. with a uniform distribution over the interval (0,1). Find the m.g.f. of X, and hence find the m.g.f. of Y, where

$$Y = \frac{X - E(X)}{\sqrt{Var(X)}}.$$

9. Let $X_i \sim N(\mu, \sigma^2)$, for i=1,2, ..., n, be n independent r.v.'s. Find the m.g.f. of $\bar{X}$. Also find the m.g.f. of

$$Y = \frac{\bar{X} - E(\bar{X})}{\sqrt{Var(\bar{X})}}.$$

10. Referring to Problem 7, find the c.f. of X and the c.f. of Y.

11. Referring to Problem 8, find the c.f.'s of X and Y.

12. Referring to Problem 9, find the c.f. of $\bar{X}$.

13. Find the c.f. of a r.v. X whose p.d.f. is given by

$$f(x) = \begin{cases} x & \text{for } 0 \le x < 1 \\ 2-x & \text{for } 1 \le x < 2 \\ 0 & \text{otherwise} \end{cases}$$

14. A r.v. X has the p.d.f.

$$f(x) = \begin{cases} \dfrac{1}{2\sqrt{X}}, & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Obtain the m.g.f. and hence calculate the mean and variance of X.

15. Show that the r.th cumluant for the distribution

$$f(x) = ce^{-cx}, \text{ where c is constant and } 0 < x < \infty .$$

is $\dfrac{1}{c^r}(r-1)!$.

16. Referring to Problem 10, find the cumulant generating function of X.