Second Lecture – Interpolation - Direct Method

2. Quadratic Interpolation: In quadratic interpolation, the value of is y estimated using the value of x through a quadratic polynomial:

$$y = B_0 + B_1 x + B_2 x^2$$

where the polynomial above is unknown and will be built from the available data in the problem. The values of y will be estimated by selecting three pairs of data points from the available data, which are:

$$(x_0, y_0), (x_1, y_1), ...(x_n, y_n)$$

where n is a positive integer. To recall, the problem is to estimate the value of y at a specific, known value of x, with the condition that $x \neq x_i$ for all values of i=0,1,2,….n Hence, we need to choose three pairs of data points, for example, $(x_a, y_a), (x_b, y_b), (x_c, y_c)$, which should be as close as possible to the desired x.

Substitute these three points into the quadratic polynomial to get three equations as follows:

$$y_a = B_0 + B_1 x_a + B_2 x_a^2$$
$$y_b = B_0 + B_1 x_b + B_2 x_b^2$$
$$y_c = B_0 + B_1 x_c + B_2 x_c^2$$

Since $(x_a, y_a), (x_b, y_b), (x_c, y_c)$ are known values from the problem, we notice that the system consists of three unknowns: B_2, B_1, B_0, and three linear equations. This system can thus be written in matrix form as follows:

$$Ax = b$$

$$\begin{bmatrix} 1 & x_a & x_a^{\ 2} \\ 1 & x_b & x_b^{\ 2} \\ 1 & x_c & x^2_{\ c} \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} y_a \\ y_b \\ y_c \end{bmatrix}$$

This matrix system represents three linear equations with three unknowns. The system is linear because the unknowns B_2, B_1, B_0 are not raised to any powers.

The system can be solved using one of the methods we discussed previously in the first semester, such as the **Gauss-Jordan method**, **Gaussian elimination method**, the **inverse matrix method**, or **LU decomposition**, along with other numerical methods like the **Jacobi method** and **Gauss-Seidel method**.

After solving, substitute the estimated values of B_2, B_1, B_0 back into the quadratic model. Then, y can be calculated by substituting the given x value into the estimated model.

Example: From the data of the previous example on determining the speed of the rocket after the time $x = 16$, use quadratic interpolation to estimate the value of y at $x = 16$.

Time in seconds	0	10	15	20	22.5	30
Speed km/min	0	227.04	362.78	517.35	602.97	901.67

Solution:
It is clear that the variable x represents time, while y represents the rocket's speed. In other words, speed y is an unknown function of time x.

Since the goal is to estimate the rocket's speed at $x = 16$, it is logical to choose the closest values to 16, which are 15 and 20. For the third value, we can choose either 10 or 22.5. Since 16 is closer to 10 than to 22.5, we prefer to choose the third value as 10. Thus, the three chosen pairs are:

$$(x_a \ , \ y_a) = (10 \ , \ 227.04)$$
$$(x_b \ , \ y_b) = (15 \ , \ 362.78)$$
$$(x_c \ , \ y_c) = (20 \ , \ 517.35)$$

By substituting these into the quadratic polynomial:

$$227.04 = B_0 + B_1(10) + B_2(10^2)$$
$$362.78 = B_0 + B_1(15) + B_2(15^2)$$
$$517.35 = B_0 + B_1(20) + B_2(20^2)$$

We can use the matrix formula :

$$\begin{bmatrix} 1 & 10 & 100 \\ 1 & 15 & 225 \\ 1 & 20 & 400 \end{bmatrix} \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \end{bmatrix}$$

Now, one of the methods for solving the system of linear equations can be used to find the unknown values. Assuming that we choose the **inverse method**, we would then construct the augmented matrix and perform the necessary transformations as follows:

$$[A : I] \overset{transformation}{\Rightarrow} [I : A^{-1}]$$

Afterward, the unknowns can be calculated using:

$$x = A^{-1} b$$

Where:

$$\left(x_a \ , \ y_a\right) = \left(10 \ , \ 227.04\right)$$
$$\left(x_b \ , \ y_b\right) = \left(15 \ , \ 362.78\right)$$
$$\left(x_c \ , \ y_c\right) = \left(20 \ , \ 517.35\right)$$

$$x = \begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix}, \quad \mathrm{b} = \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \end{bmatrix}, \quad \mathrm{A}^{-1} = \begin{bmatrix} 1 & 10 & 100 \\ 1 & 15 & 225 \\ 1 & 20 & 400 \end{bmatrix}^{-1}$$

It is worth mentioning that you can always use MATLAB to find the inverse of matrix A and also calculate the final multiplication result $\left(A^{-1}b\right)$, instead of performing the manual transformations.

$$\begin{bmatrix} 1 & 10 & 100 & : & 1 & 0 & 0 \\ 1 & 15 & 225 & : & 0 & 1 & 0 \\ 1 & 20 & 400 & : & 0 & 0 & 1 \end{bmatrix} \overset{transformation}{\Rightarrow} \begin{bmatrix} 1 & 0 & 0 & : & 6 & -8 & 3 \\ 0 & 1 & 0 & : & -0.7 & 1.2 & -0.5 \\ 0 & 0 & 1 & : & 0.02 & -0.04 & 0.02 \end{bmatrix}$$

$$\begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} = A^{-1}b$$

$$\begin{bmatrix} B_0 \\ B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 6 & -8 & 3 \\ -0.7 & 1.2 & -0.5 \\ 0.02 & -0.04 & 0.02 \end{bmatrix} \begin{bmatrix} 227.04 \\ 362.78 \\ 517.35 \end{bmatrix} = \begin{bmatrix} 12.05 \\ 17.733 \\ 0.3766 \end{bmatrix}$$

therefore, the quadratic interpolation model is as follows:

$$y = B_0 + B_1 x + B_2 x^2$$

$$y = 12.05 + 17.733x + 0.3766x^2$$

Note that the validity of the final model is limited to all values of x that satisfy the conditions $10 < x < 20$. Now, we substitute $x = 16$ into the model:

$$y = 12.05 + 17.733(16) + 0.3766(16^2)$$

$$y = 392.1876$$

Thus, the rocket's speed at 16 seconds is estimated as:

392.1876 km/min, This value is obtained using the quadratic interpolation method.

N-th Order Interpolation:
The concept of interpolation can be generalized to higher degrees (cubic, quartic, etc.) using what is known as **Order Polynomials**. Given $n+1$ pairs of data, we can construct a model of degree n (or less).

Assume that we have the data $(x_0, y_0), (x_1, y_1), ...(x_n, y_n)$, where n is a positive integer. We wish to estimate y at a specific, known value of x, with the condition that $x \neq x_i$ for all i=0,1,2,….n. We will need all the available data points in the problem to build the following n-th degree polynomial:

$$y = B_0 + B_1 x + B_1 x^2 + ... + B_n x^n$$

In the same way that we approached the case of linear and quadratic interpolation, we substitute all the given values in the problem into the above model. This results in obtaining a system of linear equations.

$$
\begin{aligned}
y_0 &= B_0 + B_1 x_0 + B_2 x_0^2 + \ldots\ldots + B_n x_0^n \\
y_1 &= B_0 + B_1 x_1 + B_2 x_1^2 + \ldots\ldots + B_n x_1 \\
. \quad & . \quad . \quad . \quad . \quad . \quad . \quad . \\
. \quad & . \quad . \quad . \quad . \quad . \quad . \quad . \\
. \quad & . \quad . \quad . \quad . \quad . \quad . \quad . \\
y_n &= B_0 + B_1 x_n + B_2 x_n^2 + \ldots\ldots + B_n x_n^n \\
.
\end{aligned}
$$

The system above contains a number of linear equations and the same number of unknowns. Therefore, it can be solved using the methods mentioned earlier for solving systems of linear equations, such as:

- Gauss-Jordan method
- Gaussian elimination method
- Inverse matrix method
- LU decomposition method

$$Ax = b$$

$$
\begin{bmatrix}
1 & x_0 & x_0^2 & \ldots\ldots & x_0^n \\
1 & x_1 & x_1^2 & \ldots\ldots & x_1^n \\
1 & x_2 & x_2^2 & \ldots\ldots & x^n{}_2 \\
. & . & . & \ldots\ldots & . \\
. & . & . & \ldots\ldots & . \\
1 & x_n & x^2{}_n & \ldots\ldots & x^n{}_n
\end{bmatrix}
\begin{bmatrix}
B_0 \\ B_1 \\ B_2 \\ . \\ . \\ B_n
\end{bmatrix}
=
\begin{bmatrix}
y_0 \\ y_1 \\ y_2 \\ . \\ . \\ y_n
\end{bmatrix}
$$

Once the system is solved and the values of the unknowns are estimated, the nth-degree interpolation model can then be applied to calculate the value of y by substituting the value of x into the estimated model. Note that the model constructed is valid for all values of x that satisfy the given conditions $x_0 < x < x_n$.

Homework:

1. For the rocket speed example data, estimate the speed of the rocket at 16 seconds using quadratic interpolation.
2. For the example data of the rocket speed, suppose we want to build an nth-degree model to estimate the speed of the rocket using the available data. Indicate the degree of the model according to the available data, write all the equations that will be used in the model, and then write the system of linear equations using the matrix formula.