and

$$P(-1 \le X < 2) = P(X = -1) + P(X = 0) + P(X = 1)$$

$$= \frac{2}{20} + \frac{3}{20} + \frac{5}{20} = \frac{10}{20} = \frac{1}{2}$$

5.4. Special Univariate Discrete Distributions

5.4.1. Bernoulli Distribution

Suppose that a trial whose outcome can be classified as either a "success" or as a "failure" is performed. let X be a r. v. taking value 1 if the outcome is success and 0 if it is failure, then X is said to be a Bernoulli random variable and its p.m.f. is given by

$$P(X = i) = p^{i}(1 - p)^{1-i}, i = 0, 1 \qquad \dots (5\cdot4\cdot1)$$

5.4.2 Binomial Distribution

Consider a series of n independent Bernoulli: trials (n is a finite integer) in which the probability of success p in each trial is constant. Then q=1-p is the probability of failure in each trial. Let X denote the number of successes in n trials. Then X can take the values 0,1,2, ... , n with non-zero probabilities, and X is said to be a binomial random variable with parameters n and p. Its p.m.f. is given by

$$P(X = k) = C(n, k)\, p^{k} q^{n-k}, k = 0, 1, \dots n \qquad \dots (5\cdot4\cdot2)$$

By applying Theorem 2.3.2 we can get

$$\sum_{k=0}^{n} P(X = k) = (p + q)^{n} = 1$$

A random variable X is said to have a binomial distribution* if it assumes only non-negative values and its p.m.f. is given by (5.4.2)

The binomial distribution serves as a good mathematical model in a number of experimental situations in which the outcome can be classified as success or failure, head or tail, boy or girl, life or death, defective or nondefective, etc.

The Bernoulli distribution is a binomial distribution with parameters 1 and p.

Example 5.4.1.

1. Consider the experiment of tossing a fair die, and we assume that the appearence of 2 or 3 as a "success". Here $S=\{1,2,3,4,5,6\}$, $X(w)=1$ if $w=2$ or 3 ; and $X(w)=0$ otherwise. Hence X has a Bernoulli distribution

$$P(X=i)=p^i(1-p)^{1-i}$$

$$=\left(\frac{1}{3}\right)^i\left(\frac{2}{3}\right)^{1-i},\ i=0,1$$

2. A family has 4 children. Assume that the birth of each sex is equally likely. Let X denote the number of boys in a family. Find the p.m.f. of X.

Solution

We have $P(B)=P(G)=\frac{1}{2}$, then $p=q=\frac{1}{2}$, $n=4$.

$$P(X=k)=C(4,k)\left(\frac{1}{2}\right)^k\left(\frac{1}{2}\right)^{4-k},\ k=0,1,2,3,4.$$

$$=C(4,k)\left(\frac{1}{2}\right)^4.$$

---

* the binomial distribution was discoverd by James Bernoulli (1654-1705) in the year 1700.

The p.m.f. of X is given by

X = k	0	1	2	3	4
P(X = k)	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Its graph is given in Fig 5.4.

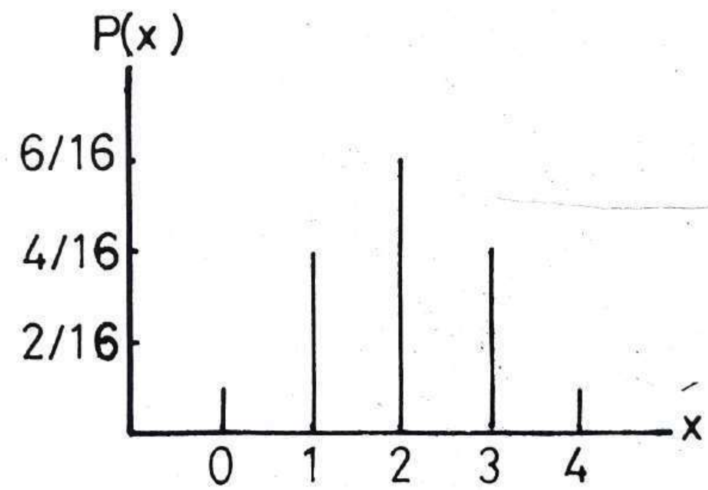

Fig. 5.4

To find the probability that the family will have at least two boys,we have

$$P(\text{at least } 2) = P(X \geq 2)$$
$$= P(X = 2) + P(X = 3) + P(X = 4)$$

$$= \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = \frac{11}{16}$$

Example 5.4.2.

In the long run 5 vessels out of 100 are sunk. If 10 vessels are out. What is the probability that

i. exactly 6 will arrive safely,
ii. at most one will sink?

Solution

i. The probability p that a vessel will arrive safely is

$$p = \frac{95}{100} = \frac{19}{20} \text{ and } q = \frac{1}{20} .$$

Let X denote the number of safety vessels. Then the probability that out of 10 vessels 6 will arrive safely is.

$$P(X = \overset{6}{k}) = C(10, 6)\left(\frac{19}{20}\right)^6\left(\frac{1}{20}\right)^4 = 0{\cdot}00096$$

ii. Let Y denote the number of sunk vessels. Then P(at most one will sink) =P(Y=0 or 1)
= P(Y=0) + P(Y=1)

$$P(Y=0) = P(X=10) = C(10,10)\left(\frac{19}{20}\right)^{10}\left(\frac{1}{20}\right)^0 = 0.5987$$

and

$$P(Y=1) = P(X=9) = C(10,9)\left(\frac{19}{20}\right)^9\left(\frac{1}{20}\right)^1 = 0.3151$$

The required probability is 0.9138

Example 5.4.3.

A study has shown that in a given university 70% of all staffs own two television sets. Find the probability that among 8 of such staffs.

a. 2 will have 2 television sets;
b. at least 6 will have 2 television sets;
c. at most 2 will have 2 television sets.

Solution

The probability that a university staff owns two television sets is

$p = 0.70$, $q=1-p=0.3$ and $n=8$

$P(X=k) = C(8,k)\ (0.7)^k\ (0.3)^{8-k}$, $k=0,1,\ \ldots\ ,8$

a. $P(X=2) = C(8,2)\ (0.7)^2\ (0.3)^6 = 0.010$

b. $P(\text{at least } 6) = P(X\geq 6) = P(X=6) + P(X=7) + P(X=8)$

$P(X=6) = C(8,6)\ (0.7)^6\ (0.3)^2 = 0.2965$

$P(X=7)\ C(8,7)\ (0.7)^7\ (0.3) = 0.1976$

$P(X=8) = C(8,8)\ (0.7)^8\ (0.3)^0 = 0.0576$

$P(X \geq 6) = 0.5517$

c. $P(\text{at most } 2) = P(x \leq 2) = P(X=0) + P(X=1) + P(X=2)$

$P(X=0) = C(8,0)\ (0.7)^0\ (0.3)^8 = 0.0001$

$P(X=1) = C(8,1)\ (0.7)\ (0.3)^7 = 0.0012$

$P(X \leq 2) = 0.0113$

5.4.3. Poisson Distribution

The Poisson distribution* appears in many natural and physical phenomena, such as

1. The number of misprints per page in a large text.
2. The number of accidents per unit of time (hour, day, week or month) on a highway.
3. The number of α - particles emitted by a radioactive substance per unit of time.
4. The number of telephone calls per unit of time received at some switchboard.
5. The number of customers entering a post office (or any goverment department) on a given period of time.

---

* The Poisson distribution is named after Simon Poisson (1781-1840), a noted French mathematician and physicist.

The Poisson distribution can be derived as a limiting case of a binomial distribution under the following conditions:

1. The number of trials n is very large.
2. The probability of success of each trial p is small enough so that np= m(say) is finite. Thus $p=\frac{m}{n}$ and $q=1-\frac{m}{n}$, where m is a positive real number. Then

$$P(X=k)=C(n,k)p^{k}q^{n-k},\ k=0,1,\ldots,n$$

$$=\frac{n(n-1)\ldots(n-k+1)}{k\,!}\left(\frac{m}{n}\right)^{k}\left(1-\frac{m}{n}\right)^{n-k}$$

$$=\frac{n^{k}\cdot 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)\left(\frac{m}{n}\right)^{k}\left(1-\frac{m}{n}\right)^{n}}{k\,!\left(1-\frac{m}{n}\right)^{k}}$$

$$=\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(-\frac{k-1}{n}\right)\ m^{k}\left(1-\frac{m}{n}\right)^{n}}{k\,!\left(1-\frac{m}{n}\right)^{k}}$$

If we let $n \to \infty$ while k and m remain constants, we find that

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\ldots\left(1-\frac{k-1}{n}\right)\to 1$$

$$\left(1-\frac{m}{n}\right)^{k}\to 1$$

$$\left(1-\frac{m}{n}\right)^{n}\to e^{-m}$$

Hence (5.4.2) is reduced to the form

$$P(X = k) = \frac{e^{-m} . m^k}{k!}, k = 0, 1, 2, \ldots \qquad \ldots\ldots(5.4.3)$$

A random variable having p.m.f given by (5. 4. 3) is said to have a Poission distribution with parameter m. The quantity m can be interpreted as the mean rate of occurrences of an event occurring in a unit of time (may be a page, second, minute, hour, etc.)

Recall that the series

$$1 + m + \frac{m^2}{2!} + \ldots = \sum_{k=0}^{\infty} \frac{m^k}{k!}$$

converges, for all values of m, to e^m, therefore

$$\sum_{k=0}^{\infty} P(X = k) = \sum_{k=0}^{\infty} \frac{e^{-m} m^k}{k!} = 1$$

Table (1) in Appendix gives e^{-m} for some values of m.

Example 5.4.4

Suppose 3% of the items made by a factory are defective. A sample of 100 items is drawn at random, what is the probability that it will contain exactly 2 defective items,? Using (a) binomial distribution, (b) Poisson distribution.

Solution :

Here P=0.03, q=0.97 and n=100, then.

a. $P(X=2) = C(100,2)\ (0.03)^2 (0.97)^{98}$
$= 0.2251$

b. $m = np$
$= 100\ (0.03) = 3$

$$P(X=2) = \frac{e^{-3} . 3^2}{2!} = 0.0497\ (4.5) = 0.2240$$

Example 5.4.5.

An office switchboard receives phone calls at the rate of 2 calls per minute on average. Find the probability that the office will receive at least 3 phone calls.

Solution

we have m=2, then

$$P(X \geq 3) = 1 - \{P(X=0) + P(X=1) + P(X=2)\}$$

$$P(X=0) = \frac{e^{-2} \cdot 2^0}{0!} = 0{\cdot}1353(1) = 0{\cdot}1353$$

$$P(X=1) = \frac{e^{-2} \cdot 2}{1!} = 0{\cdot}1353(2) = 0{\cdot}2706$$

$$P(X=2) = \frac{e^{-2} \cdot 2^2}{2!} = \frac{0{\cdot}1353(4)}{2} = 0{\cdot}2706$$

The required probability is.
$P(X \geq 3) = 1\text{-}0.6765 = 0.3235$

Example 5.4.6

If the number of accidents on the highway between Mosul and Baghdad between 9P. M. and 10 P. M. on a Friday is a r.v. having a Poisson distribution with m=0.5, find the probability that there will be

1. fewer than 3 accidents,
2. anywhere from 2 to 6 accidents, inclusive on this highway between 9 P.M. and 10 P.M. on a Friday.

Solution

1. $P(X < 3) = P(X = 0) + P(X = 1) + P(X = 2)$

$$= \frac{e^{-0.5}(0.5)^0}{0!} + \frac{e^{-0.5}(0.5)}{1!} + \frac{e^{-0.5}(0.5)^2}{2}$$

$$= e^{-0.5}(1.625) = 0.6065(1.625)$$
$$= 0.986$$

2. $P(2 \le X \le 6) = \sum_{2}^{6} \frac{e^{-0.5}(0.5)^k}{k!} = e^{-0.5}\,\Sigma \frac{1}{2^k\,k!}$

$$= 0.6065\left[\frac{1}{8} + \frac{1}{48} + \frac{1}{384} + \frac{1}{3840} + \frac{1}{46080}\right]$$

$$= 0.6065(0.146) = 0.0885$$

Example 5.4.7.

1. A company has 450 ships. If on average one ship is lost in the sea for every 150 ships, what is the probability that at least two ships are lost to the company?

Solution :

The average number of ships lost in the sea are m=3. Then

$$P(X \ge 2) = 1 - \{P(X = 0) + P(X = 1)\}$$

$$= 1 - \left[\frac{e^{-3}\,3^0}{0!} + \frac{e^{-3}\,3}{1}\right] = 1 - 4e^{-3}$$

$$= 1 - 4(0.0497)$$
$$= 0.8012$$

2. The number of traffic accidents of a city in each week is assumed to be a Poisson variate with parameter 1.2. What is the probability that there will be at most one accident in February?

Solution :

In February there are 4 weeks, therefore the average number of accidents is 4(1.2) = 4.8.

$P(X \leq 1) = P(X=0) + P(X=1)$

$$= \frac{e^{-4\cdot8}(4\cdot8)^0}{0!} + \frac{e^{4\cdot8}(4\cdot8)}{1!}$$

$$= (0\cdot0082)(5\cdot8) = 0\cdot04756$$

5.4.4. Geometric Distribution

Suppose that independent trials, each having a probability p of being a success, are performed until the first success occurs. If X denotes the number of trials required, then

$$P(X=n) = p\,q^{n-1} \quad , n=1,2, \ldots.. \quad p+q=1 \qquad (5.4.4)$$

A r.v. X having p.m.f. given by (5.4.4) is said to have a geometric distribution (or Pascal distribution) with parameter p.

Recall that the series

$$1 + q + q^2 + \ldots = \sum_{n=0}^{\infty} q^n$$

converges for $q < 1$ to $\frac{1}{1-q}$, therefore

$$\sum_{n=1}^{\infty} pq^{n-1} = \frac{p}{1-q} = 1$$

Note :

If X denotes the number of failures preceding the first success, then

$$P(X=n) = pq^n \quad , n=0,1,2,3, \ldots \qquad \ldots\ldots (5.4.5)$$

Example 5.4.8

1. If the probability that a person will believe a rumor about the retirement of a certain politician. is $\frac{1}{3}$,

   Then the probability that the fifth. person to hear the rumor will be the first one to believe it is

$$P(X=5) = \frac{1}{3}\left(\frac{2}{3}\right)^{5-1}$$

$$= \frac{1}{3} \cdot \frac{16}{81} = \frac{16}{243}$$

2. If the probability of the birth of a child with a defective heart, in a special hospital is 0.07, then the probability that the 10 th child born in the hospital is the first one which has a defective heart is .
   $P(X=10) = (0.93)^9 (0.07) = 0.036428$

5.4.5. Hypergeometric Distribution

When the population consists of N objects (N is finite) of which m are of type I and (N-m) are of type II. A sample of size n is drawn at random from this population without replacement. If we let X be the number of type I in the sample, then.

$$P(X=k) = \frac{C(m,k).C(N-m,n-k)}{C(N,n)}, k = 0,1,\ldots,$$

$$k = 0, 1, \ldots\ldots, \min(m,n). \qquad \ldots(5{\cdot}4{\cdot}6)$$

A r.v. X is said to have a hypergeometric distribution if its p.m.f. is given by (5.4.6)

We have $\sum\limits_{k=0} P(X=k) = 1$ by Theorem 2.3.3.

Example 5.4.9.

A lot of 20 radio tubes, 8 of which are defective. A sample of 5 items is selected from it. Find the probability that the sample will contain.

i. one defective radio tube,
ii. at least 4 defective radio tubes.

Solution

i. $$P(X = 1) = \frac{C(8,1).C(12,4)}{C(20,5)} = \frac{8(595)}{15504}$$

$$= 0{\cdot}2554.$$

ii. $$P(X \geq 4) = \frac{C(8,4)C(12,1)}{C(20,5)} + \frac{C(8,5)C(12,0)}{C(20,5)}$$

$$= \frac{70(12) + 56(1)}{15504} = 0{\cdot}0577.$$

5.4.6 Negative Binomial Distribution

In the geometric disribution we are interested in the first success. If we are interested in the k-th success instead of the first, in this case we apply the negative binomial distribution which can be explained as follows:

Suppose we have n independent Bernoulli trials, we can say that the probability of getting (k-1) success in (n-1) trials is

$$C(n-1, k-1)\, p^{k-1} q^{n-k}$$