# Testing a Population Mean: Known and Unknown $\sigma$

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## Introduction

In this lecture, we study how to test a hypothesis about a population mean $\mu$. Two situations are common:

1. The population standard deviation $\sigma$ is **known** (or the sample is large, $n \geq 30$). We use the **Z-test**.
2. The population standard deviation $\sigma$ is **unknown** and the sample is small ($n < 30$). We use the **one-sample t-test**.

The logic of both tests comes from the **Central Limit Theorem (CLT)**:

$$\bar{X} \approx N\left(\mu, \frac{\sigma^2}{n}\right)$$

which means that the sampling distribution of $\bar{X}$ is approximately normal when $n$ is large, even if the population itself is not perfectly normal.

## Why Do We Test a Population Mean?

We test the mean $\mu$ to check if a process, product, or outcome meets a promised or target value. The test protects decisions (approve a batch, change a policy, start a treatment) from being driven by random sampling noise.

**Quality control (factory).** A machine should fill bags at $\mu_0 = 1000$ g. One sample of $n$ bags gives a sample mean $\bar{x}$. We test

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu \neq \mu_0.$$

Rejecting $H_0$ means the machine is off target and needs adjustment.

**Health benchmark.** A clinic wants to see if a diet reduces average cholesterol below 200 mg/dL:

$$H_0 : \mu = 200 \quad \text{vs} \quad H_1 : \mu < 200 \quad \text{(left-tailed)}.$$

Rejecting $H_0$ suggests the diet truly lowers the mean.

**Calibration / bias check.** A new thermometer claims to read true temperature on average. Using a reference bath at $\mu_0$, test

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu \neq \mu_0.$$

Rejecting $H_0$ means there is systematic bias.

**Program evaluation.** A training program targets an average exam score of 70. After training, a class has mean $\bar{x}$. We test

$$H_0 : \mu = 70 \quad \text{vs} \quad H_1 : \mu > 70 \quad \text{(right-tailed)}.$$

Rejecting $H_0$ supports that the program exceeds the target.

### When to use one- vs two-sided?

Use **two-sided** ($H_1 : \mu \neq \mu_0$) when any deviation is important. Use **one-sided** ($H_1 : \mu > \mu_0$ or $H_1 : \mu < \mu_0$) only when a deviation in one direction matters and this direction is set *before* looking at the data.

# 1 Case 1: $\sigma$ known (Z-test)

### Formulation

We test the null and alternative hypotheses:

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \begin{cases} \mu \neq \mu_0 & \text{(two-sided)} \\ \mu > \mu_0 & \text{(right-tailed)} \\ \mu < \mu_0 & \text{(left-tailed)} \end{cases}$$

The test statistic is

$$Z = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

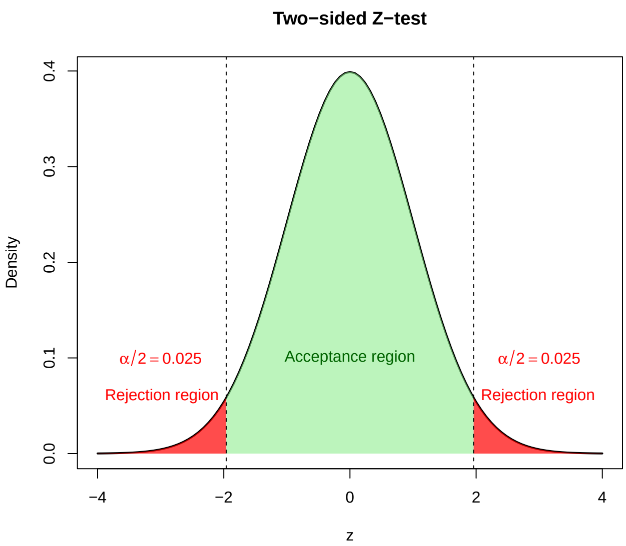
Under $H_0$, $Z$ follows the standard normal distribution $N(0, 1)$.

### Decision rule

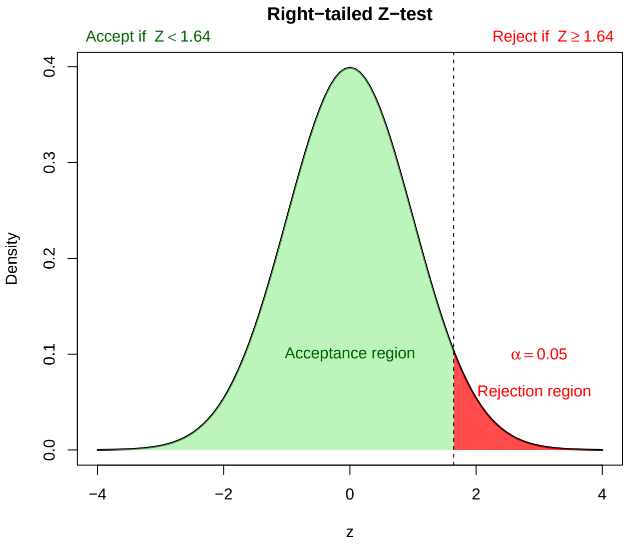
$$\text{Reject } H_0 \text{ if } \begin{cases} |Z| > z_{\alpha/2}, & \text{two-sided test} \\ Z > z_{\alpha}, & \text{right-tailed test} \\ Z < -z_{\alpha}, & \text{left-tailed test} \end{cases}$$

## R Code for Visualization

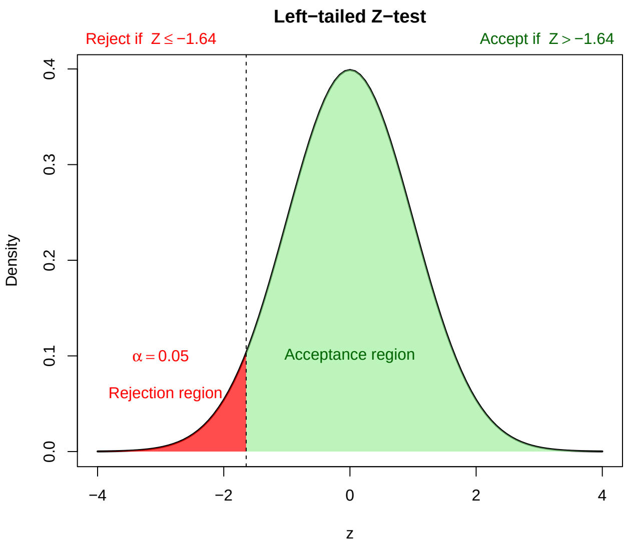
#### Two-sided Z-test or t-test visualization




#### Right-tailed test




**Left-tailed test**




## Example (Hand Computation)

A company claims that the mean weight of sugar packets is 1 kg. From a large shipment, you take $n = 36$ packets, and find

$$\bar{x} = 0.97 \text{ kg}, \quad \sigma = 0.06 \text{ kg}.$$

Test at $\alpha = 0.05$ if the mean differs from 1 kg.

**Solution:**

$$Z = \frac{0.97 - 1.00}{0.06/\sqrt{36}} = \frac{-0.03}{0.01} = -3.0.$$

Critical values: $z_{0.025} = \pm 1.96$.

Since $|Z| = 3.0 > 1.96$, we **reject** $H_0$. The mean weight is significantly different from 1 kg.