Lecture 3: Numerical Analysis (2)

1. Linear Interpolation Using Newton's Polynomial for Divided Differences:

To recall, we are given a set of value pairs: $(x_0, y_0), (x_1, y_1) \dots, (x_n, y_n)$, where is n a positive integer. Our mathematical problem is to estimate the value of y for a given x, with the condition that $x \neq x_i$ for all i=0,1,2,….n

The linear interpolation polynomial using Newton's method for divided differences is constructed as follows:

$$y = B_0 + B_1(x - x_0) \dots\dots\dots\dots\dots\dots\dots(1)$$

To estimate B_0 & B_1 using the above equation, we need to choose two pairs from the available data: $(x_a, y_a), (x_b, y_b)$. This selection is done in the same way as discussed in direct linear interpolation, with the same conditions (refer to the first lecture for details).

Important note: We will re-encode the first selected pair (x_a, y_a) as (x_0, y_0), and similarly, (x_b, y_b) will be re-encoded as (x_1, y_1), and so on. The purpose of this re-coding is to simplify the representation of the equations in Newton's method for divided differences. This does not mean we selected the first two pairs from the original data; we are just renaming the selected pairs using zero-based $0,1,2,\dots$ indexing instead of the original labels $a, b, c, \dots$.

Thus, in the equation (1) above, the first pair selected corresponds to (x_0, y_0), and the second pair corresponds to (x_1, y_1), Now, we substitute the selected pairs into equation (1) as follows:

$$y_0 = B_0 + B_1(x_0 - x_0)$$

$$\Rightarrow B_0 = y_0 \text{..........}(2)$$

Similarly, for the second pair (x_1, y_1), we have:

$$y_1 = B_0 + B_1 (x_1 - x_0) \text{............}(3)$$

$B_0 = y_0$ From equation (2), substitute into equation (3):

$$y_1 = y_0 + B_1 (x_1 - x_0)$$

$$\Rightarrow B_1 = \frac{y_{1-} \; y_0}{x_1 - x_0} \text{........}(4)$$

The term B_0 is called the 0^{th} divided difference and is denoted as $f[x_0]$, while B_1 is the 1^{st} first divided difference and is denoted as $f[x_0, x_1]$. Therefore:

$$B_0 = y_0 = f[x_0] \text{........}(2)$$

$$B_1 = \frac{y_{1-} \; y_0}{x_1 - x_0} = f[x_0, x_1] \text{.......}(4)$$

After calculating the values of B_0 & B_1 the divided differences, we can apply the formula to estimate y at any x value within the range (x_b, y_b) as determined by the initial selection.

2. Quadratic Interpolation Using Newton's Polynomial for Divided Differences:

We face the same problem as before, but now we build the quadratic interpolation polynomial using Newton's method for divided differences as follows:

$$y = B_0 + B_1(x - x_0) + B_2(x - x_0)(x - x_1) \text{..........(5)}$$

To estimate B_0, B_1, B_2 using the above equation, we need to select three pairs of data: $(x_a, y_a), (x_b, y_b), (x_C, y_C)$. This selection follows the same conditions as we discussed in quadratic interpolation in lecture 2 (refer to lecture 2 for more details).

After selecting and re-encoding the pairs similarly to the linear interpolation method, we can proceed to calculate the required divided differences.

Now, we substitute the selected pairs into equation (5) as follows:

$$(x_0 \ , \ y_0) = (x_a \ , \ y_a), \quad (x_1 \ , \ y_1) = (x_b \ , \ y_b), \quad (x_2 \ , \ y_2) = (x_c \ , \ y_c)$$

were selected. Then, we substitute the three pairs into equation (5) as follows:

For the first pair $(x_0 \ , \ y_0)$:

:

$$y_0 = B_0 + B_1(x_0 - x_0) + B_2(x_0 - x_0)(x_0 - x_1)$$

$$\Rightarrow B_0 = y_0 = f[x_0] \text{.................(6)}$$

Similarly, for the second pair $(x_1 \ , \ y_1)$

$$y_1 = B_0 + B_1(x_1 - x_0) + B_2(x_1 - x_0)(x_1 - x_1) \text{...........(7)}$$

Since $B_0 = y_0$ **from equation (6) above, we substitute into equation (7) as follows:**

$$\Rightarrow y_1 = y_0 + B_1(x_1 - x_0)$$

$$\Rightarrow B_1 = \frac{y_{1-}\ y_0}{x_1 - x_0} = f[x_0, \mathrm{x}_1] \text{........(8)}$$

Similarly, for the third pair $(x_2\ ,\ y_2)$:

$$y_2 = B_0 + B_1(x_2 - x_0) + B_2(x_2 - x_0)(x_2 - x_1) \text{............(9)}$$

Substituting equations (6) and (8) into equation (9) gives:

$$y_2 = y_0 + \frac{y_{1-}\ y_0}{x_1 - x_0}(x_2 - x_0) + B_2(x_2 - x_0)(x_2 - x_1)$$

$$\Rightarrow B_2 = \frac{\dfrac{y_{2-}\ y_0}{x_2 - x_1} - \dfrac{y_1 - y_0}{x_1 - x_0}}{(x_2 - x_1)}$$

Or it can be expressed as:

$$\Rightarrow B_2 = \frac{\frac{y_{2-} y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{(x_2 - x_0)}$$

In its form shown in equation (10), it is called B_2 the 2^{th} second divided difference, denoted as $f[x_0, \mathrm{x}_1, \mathrm{x}_2]$. That is:

$$B_2 = \frac{\frac{y_{2-} y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{(x_2 - x_0)} = \frac{f[\mathrm{x}_1, \mathrm{x}_2] - f[x_0, \mathrm{x}_1]}{(x_2 - x_0)} = f[x_0, \mathrm{x}_1, \mathrm{x}_2]$$

………(10)

Where $f[x_0, \mathrm{x}_1] = \frac{y_{1-} y_0}{x_1 - x_0}$ is the first divided difference between (x_0, x_1)

and $f[\mathrm{x}_1, \mathrm{x}_2] = \frac{y_{2-} y_1}{x_2 - x_1}$ is the first divided difference between (x_1, x_2) so

$$B_0 = f[x_0], \quad B_1 = \ f[x_0, \mathrm{x}_1], \ B_2 = \ f[x_0, \mathrm{x}_1, \mathrm{x}_2]$$

After calculating the values B_0, B_1, B_2 of the divided differences, equation (5) is applied to the value of x to estimate the value of y, where the validity of equation (5) is limited to values of x that fall within the range of the three selected points (x_a, x_b, x_c) that were initially chosen.

Example:

The table below shows the values of the natural logarithm of some real numbers, according to the indicator about each of them:

7	6.3	6	5.5	4.5	4	x
1.945910	1.840550	1.791759	1.704748	1.504077	1.386294	y=ln(x)

It is required to estimate the value of the natural logarithm of $x = 5$ using both **linear interpolation** and **quadratic interpolation** using the Newton polynomial method for divided

Solution:

. **Linear Interpolation:** The linear interpolation function will be as follows:

$$y = B_0 + B_1(x - x_0)$$

- Notice that the two closest values to $x = 5$ are $(x_a \ , \ y_a) = (4.5, 1.504077), (x_b \ , \ y_b) = (5.5, 1.704748)$ .Therefore, we re-label the points accordingly.

$$(x_0 \ , \ y_0) = (x_a \ , \ y_a) = (4.5, 1.504077)$$

$$(x_1 \ , \ y_1) = (x_b \ , \ y_b) = (5.5, 1.704748)$$

By applying equations (2) and (4) to calculate the 0^{th} and first divided differences, we then substitute the values into the linear interpolation formula.

$$B_1 = f\left[x_0, \mathrm{x}_1\right] = \frac{y_{1-}\ y_0}{x_1 - x_0} = \frac{1.704748 - 1.504077}{5.5 - 4.5} = 0.200671$$

By substituting the values B_0, B_1 into the linear interpolation function:

$$y = 1.504077 + 0.200671(x - 4.5)$$

The validity of the final equation is restricted to the interval (4.5,5.5). Therefore, the estimated value of the natural logarithm of the number 5 is as follows:

$$y = 1.504077 + 0.200671(5 - 4.5) = 1.604413$$

2. **Quadratic Interpolation:**

For quadratic interpolation,:

We need to select three pairs according to the conditions mentioned in Lecture 2. Notice that:

$$(x_a \ \ , \ y_a) = (4.5, 1.504077)$$

$$(x_b \ \ , \ y_b) = (5.5, 1.704748)$$

$$(x_c \ \ , \ y_c) = (6,\ 1.791759)$$

Were-label as follows:

$$(x_0 \ \ , \ y_0) = (x_a \ \ , \ y_a) = (4.5, 1.504077)$$

$$(x_1 \ \ , \ y_1) = (x_b \ \ , \ y_b) = (5.5, 1.704748)$$

$$(x_2 \ \ , \ y_2) = (x_c \ \ , \ y_c) = (6,\ 1.791759)$$

It is also possible to select the pair corresponding to x=4 instead of the pair corresponding to. x=6 Now, by applying equations (6), (8), and (10), we calculate the zero, first, and second divided difference:

$$B_0 = f\left[x_0\right] = y_0 = 1.504077$$

$$B_1 = f\left[x_0, \mathrm{x}_1\right] = \frac{y_{1-} \; y_0}{x_1 - x_0} = \frac{1.704748 - 1.504077}{5.5 - 4.5} = 0.200671$$

$$B_2 = f\left[x_0, \mathrm{x}_1, \mathrm{x}_2\right] = \frac{f\left[\mathrm{x}_1, \mathrm{x}_2\right] - f\left[x_0, \mathrm{x}_1\right]}{\left(x_2 - x_0\right)}$$

Where as:

$$f\left[x_0, \mathrm{x}_1\right] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{1.704748 - 1.504077}{5.5 - 4.5} = 0.200671$$

$$f\left[\mathrm{x}_1, \mathrm{x}_2\right] = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1.791759 - 1.704748}{6 - 5.5} = 0.174022$$

By substituting into the second divided difference equation:

$$B_2 = \frac{0.174022 - 0.200671}{6 - 4.5} = -0.017766$$

Therefore, the interpolation function is as follows:

$$y = B_0 + (x - x_0)B_1 + (x - x_0)(x - x_1)B_2$$

$$y = 1.504077 + 0.200671(x - 4.5) - 0.017766(x - 4.5)(x - 5.5)$$

The validity is also restricted to values of xxx within the range (4.5,6). Now, by substituting $x = 5$, we obtain:

$$y = 1.504077 + 0.200671(5 - 4.5) - 0.017766(5 - 4.5)(5 - 5.5)$$

$$y = 1.608854$$

Homework:

For the example data above, estimate the value of the natural logarithm of x=5.8 using the dividing differences method in quadratic interpolation only.