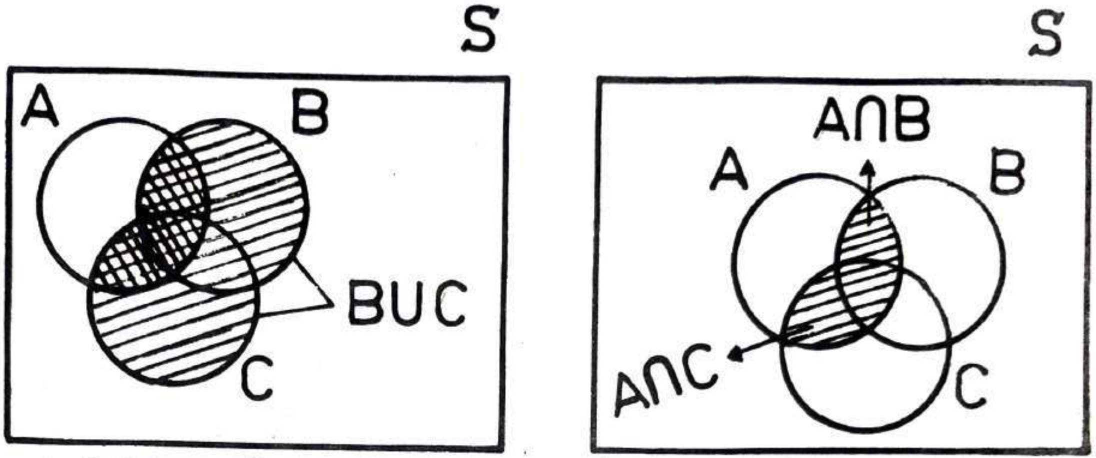

$A\cap(B\cup C)$ is shaded $(A\cap B)\cup(A\cap C)$ is shaded

Fig. 1.2

1.3 Sequences and Limits

We have already defined union and intersection of two sets; these definitions extend to more than two sets. To distinguish between the sets in a collection of subsets of S we assign names to them in the form of subscripts. Let I denote a catalogue of names, or indicies, I is also called an index set. For example if we have four sets, then our index set $I=\{1,2,3,4\}$.

Definition 1.3.1

Let I be an index set and $\{A_r : r \in I\}$ be a collection of subsets of S indexed by I, which is called a ***sequence of sets***.

The *union* of an arbitrary number of sets is given by

$$\bigcup_{r\in I} A_r = \{w : w \in S\ ;\ w \in A_r \textit{ for some } r \in I\}$$

Hence

$$w \in \bigcup_{r\in I} A_r \text{ if and only if } w \in A_r \textit{ for some } r \in I \qquad (1.3.1)$$

and

$$w \notin \bigcup_{r \in I} A_r \text{ if and only if } w \notin A_r \textit{ for all } r \in I \qquad (1.3.2)$$

The *intersection* of any arbitrary number of sets is given by

$$\bigcap_{r \in I} A_r = \{w : w \in S \,;\, w \in A_r \textit{ for all } r \in I\}$$

Hence

$$w \in \bigcap_{r \in I} A_r \text{ if and only if } w \in A_r \textit{ for all } r \in I \qquad (1.3.3)$$

and

$$w \notin \bigcap_{r \in I} A_r \text{ if and only if } w \notin A_r \textit{ for some } r \in I \qquad (1.3.4)$$

If I is finite, we have *finite union* and *finite intersection*. If it is countably infinite and is given by 1,2,3, ..., we have *countable union* and *countable intersection* which are denoted by $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$, respectively.

Example 1.3.1

Let $I = \{1,2,\dots,N\}$, N is finite. Define
$A_i = \{ x : 1 < x < 1+1/i\}$, $i \in I$.

Then we have

$$\bigcup_{i \in I} A_i = A_1 = \{x : 1 < x < 2\}$$

and

$$\bigcap_{i \in I} A_i = A_N = \{x : 1 < x < 1+1/N\}$$

Theorem 1.3.1- De Morgan's Theorem

Let I be an index set and $\{A_r : r \in I\}$ a sequence of subsets of S indexed by I. Then

1. $$\left(\bigcup_{i \in I} Ai\right)' = \bigcap_{i \in I} Ai' \qquad (1.3.5)$$

2. $$\Big(\bigcap_{i\epsilon I} A_i\Big)' = \bigcup_{i\epsilon I} A_i' \qquad (1.3.6)$$

To prove (1.3.5), let

$$w \in \Big(\bigcup_{i=1} Ai\Big)' \Longleftrightarrow w \in S \text{ and } w \notin \bigcup_{i\epsilon I} A_i$$

$$\Longleftrightarrow w \in S \text{ and } w \notin A_i \text{ for all } i \in I \qquad \text{by (1.3.2)}$$
$$\Longleftrightarrow w \in A_i \text{ for all } i \in I$$
$$\Longleftrightarrow w \in \bigcap_{i\epsilon I} A_i' \qquad \text{by (1.3.3)}$$

We can prove (1.3.6) in the same way.

Lemma 1.3.1

Given a class $\{A_i,\ i=1,2, \ldots\ldots, n\}$ of n sets, there exists a class $\{B_i,\ i= 1, \ldots\ldots, n\}$ of disjoint sets such that

$$\bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i$$

(The sets B_1, B_2, ………… are said to be disjoint or mutullay exclusive if $B_i \cap B_j = \phi$ for every $i \neq j$)

We can prove lemma 1.3.1 by induction. For n=2 we have $A_1 \cup A_2 = A_1 \cup (A_1' \cap A_2)$ which is a union of disjoint sets. Taking $B_1 = A_1$ and $B_2 = A_1' \cap A_2$, therefore

$$A_1 \quad A_2 = B_1 + B_2.$$

Suppose it is true for $m \geq 2$, i.e.

$$\bigcup_{i=1}^{m} A_i = \sum_{i=1}^{m} B_i.$$

To show it is true for n = m+1, we have

$$\bigcup_{i=1}^{m+1} A_i = (\bigcup_{i=1}^{m} A_i) \bigcup \ A_{m+1}$$

$= (\bigcup_{i=1}^{m} A_i) \bigcup [(\bigcup_{i=1}^{m} A_i)' \bigcap A_{m+1}]$ which is a union of disjoint sets. Taking $B_{m+1} = (\bigcup_{i=1}^{m} A_i)' \bigcap A_{m+1}$, so we have

$$\bigcup_{i=1}^{m+1} A_i = (\sum_{i=1}^{m} B_i) \bigcup \ B_{m+1}$$

$$= \sum_{i=1}^{m+1} B_i$$

Hence the lemma follows.

Definition 1.3.2

A sequence of sets $\{ A_n \} : n = 1,2, \ldots \}$ is said to be increasing (or non-decreasing) if $A_n \subseteq A_{n+1}$ for each n. If $A_n \supseteq A_{n+1}$ for each n, then the sequence {An} is said to be *decreasing* (or non-increasing). *A monotone* sequence is one which is either increasing or decreasing.

The set of all points which belong to infintely many A_n's is called the superior limit of A_n, and is denoted by lim sup A_n. The set of all points which belong to almost all A_n (except for finite number of sets is called the inferior limit of A_n.

We have

$$\limsup A_n = \lim A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$$

and

$$\liminf A_n = \underline{\mathit{lim}} \ A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$$

Also, we have

$$\underline{\mathit{lim}} \ A_n \subseteq \overline{\lim} \ A_n.$$

A sequence $\{A_n\}$ is said to be convergent to A if $\overline{\lim}\ A_n = \underline{\lim}\ A_n$ Note that any monotone sequence converges to a limit since.

1. If $\{A_n\}$ is increasing, then $\bigcup\limits_{k=n}^{\infty} A_k = \bigcup\limits_{K=1}^{\infty} A_K$ and $\bigcap\limits_{k=n}^{\infty} A_k = A_n$ for all n, Therefore

   $$\overline{\lim}\ A_n = \underline{\lim}\ A_n = \bigcup_{k=1}^{\infty} A_k$$

2. If $\{A_n\}$ is decreasing, then $\bigcup\limits_{k=n}^{\infty} A_k = A_n$ and $\bigcap\limits_{k=n}^{\infty} A_k = \bigcap\limits_{k=1}^{\infty} A_k$ for all n. There foro $\overline{\lim}\ A_n = \underline{\lim}\ A_n = \bigcap\limits_{k=1}^{\infty} A_k$

Example 1.3.2

Let S be the set of nonnegative real numbers. Define

$$A_n = \left[0, 1 - \frac{1}{n}\right] = \left\{x : x \in S, 0 \le x \le 1 - \frac{1}{n}\right\}, n = 1,2,\ldots$$

we have

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left[0, 1 - \frac{1}{n}\right] = [0, 1) = \{x : 0 \le x < 1\}$$

and

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} \left[0, 1 - \frac{1}{n}\right] = \{0\}$$

To verify the De Morgan's Theorem, we have

$$\left(\bigcup_{n=1}^{\infty} A_n\right)' = [0, 1)' = [1, \infty) = \{x : 1 \le x < \infty\}$$

$$\bigcap_{n=1}^{\infty} A_n' = \bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, \infty\right) = [1, \infty)$$

(Notice that $x > 1 - \dfrac{1}{n}$ for all n= 1,2, ... $\Longleftrightarrow x \ge 1$)