Testing a Population Mean: Unknown σ

Dr. Zaid T. Al-Khaledi
Department of Statistics and Informatics
University of Mosul

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1 Case 2: σ unknown (t-test)

Formulation

When σ is unknown and the sample is small, we estimate it by s and use the t-statistic:

$$t = \frac{\bar{X} - \mu_0}{s/\sqrt{n}},$$

which follows a t distribution with $n - 1$ degrees of freedom under H_0.

$$H_0 : \mu = \mu_0, \quad H_1 : \begin{cases} \mu \neq \mu_0 & \text{(two-sided)} \\ \mu > \mu_0 & \text{(right-tailed)} \\ \mu < \mu_0 & \text{(left-tailed).} \end{cases}$$

Decision rule

$$\text{Reject } H_0 \text{ if } \begin{cases} |t| > t_{\alpha/2,n-1}, & \text{two-sided test} \\ t > t_{\alpha,n-1}, & \text{right-tailed test} \\ t < -t_{\alpha,n-1}, & \text{left-tailed test} \end{cases}$$

where α is the level of significance (usually $\alpha = 0.05$), and $n - 1$ is the degrees of freedom ($df = n - 1$).

Example (Hand Computation)

A nutritionist wants to check if a new diet changes average cholesterol level from the known baseline $\mu_0 = 200$ mg/dL. A sample of $n = 10$ participants gives:

$$\bar{x} = 190, \quad s = 12.$$

Use $\alpha = 0.05$.

$$t = \frac{190 - 200}{12/\sqrt{10}} = \frac{-10}{3.79} = -2.64.$$

Degrees of freedom: $df = 9$.

Critical values for $\alpha = 0.05$ (two-sided): ± 2.262.

Since $|t| = 2.64 > 2.262$, we **reject** H_0. The mean cholesterol level has significantly changed.

Example: Large n with Unknown σ (Use s)

Even when σ is unknown, the CLT tells us that for large n the sampling distribution of $\bar{X}$ is close to normal. The standard, modern approach is to use the one-sample **t-test** with s; for large n the t and z results are almost the same.

Scenario. A device is supposed to deliver $\mu_0 = 100$ units. A sample of $n = 50$ readings gives

$$\bar{x} = 98.7, \quad s = 5.2.$$

Test at $\alpha = 0.05$.

Two-sided test. $H_0 : \mu = 100$ vs $H_1 : \mu \neq 100$.

$$t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{98.7 - 100}{5.2/\sqrt{50}} = -1.768 \quad (\text{df} = 49).$$

Critical value: $t_{0.025,49} \approx 2.009$. Since $|t| = 1.768 < 2.009$, we **fail to reject** H_0. *Two-sided p-value* ≈ 0.083.

Because σ is unknown, we usually use the t-test. However, with $n = 50$ the t and normal critical values are close ($t_{0.025,49} \approx 2.009$ vs. $z_{0.025} = 1.96$).

In both cases, the test statistic $|t| = |Z| = 1.768$ is below the corresponding critical value, so we **fail to reject** H_0.

Left-tailed test. $H_0 : \mu = 100$ vs. $H_1 : \mu < 100$. Critical value (left tail): $t_{0.05,49} \approx -1.676$. Since $t = -1.768 < -1.676$, we **reject** H_0 in the left-tailed case. *One-sided p-value* ≈ 0.042.

Takeaway. For large n, using t with s or using z with s gives very similar decisions; the t distribution is slightly wider, but the difference is small when n is large. It is safe and standard to always use the **t-test with** s when σ is unknown.

Interpretation

If σ is unknown, we must assume the population is approximately normal (checked using Q-Q plots or Shapiro-Wilk test). For large n, the t-test and Z-test give nearly the same result.

Summary of the Procedure

Step	**Description**
1. State hypotheses	$H_0 : \mu = \mu_0$, H_1 : one- or two-sided
2. Check assumptions	Normality (if small n), independence
3. Compute statistic	$Z = (\bar{X} - \mu_0)/(\sigma/\sqrt{n})$ or $t = (\bar{X} - \mu_0)/(s/\sqrt{n})$
4. Find critical value	From normal or t tables for given α
5. Decision rule	Reject H_0 if test statistic in rejection region
6. Interpretation	Write conclusion in simple words

Homework: One-Mean Z and t Tests

Use $\alpha = 0.05$ for all problems. Decide the hypotheses (one- or two-sided), the appropriate test, compute the test statistic and critical value(s), and give a clear conclusion in context.

Q1 (Factory quality)

A filling line is supposed to deliver **500 mL** per IV bag. An engineering validation study previously measured the process standard deviation as $\sigma = 4$ **mL** (assume the process remains stable). From today's production, a simple random sample of $n = 40$ bags has sample mean $\bar{x} = 498.9$ **mL**. Is the process on target?

Q2 (Sports performance)

A sprint coach evaluates a new warm-up routine for 100 m. From a training group, the following summaries are obtained:

$$n = 25, \qquad \bar{x} = 11.82, \qquad \sum_{i=1}^{n} x_i^2 = 3500.31.$$

Based on these data, is there evidence that the routine changes the average time from **12.00**?

Q3 (Medical–HbA1c)

A clinic reviews a lifestyle program intended to improve glycemic control. After the program, a simple random sample of patients shows the following HbA1c (%) values:

$$6.9,\ 6.7,\ 6.8,\ 7.3,\ 6.5,\ 6.6,\ 6.8,\ 6.4,\ 7.1,\ 6.9,\ 6.7,\ 6.8,\ 6.6,\ 6.5,\ 6.8,\ 7.0.$$

Using the clinical guideline value **7.0%** as a benchmark, is there evidence that the program achieves a different average HbA1c?

Candidate Critical Values

Label	Value	Label	Value	Label	Value
$z_{0.10}$	1.282	$z_{0.05}$	1.645	$z_{0.025}$	1.960
$z_{0.01}$	2.326	$z_{0.005}$	2.576	$t_{0.05,\,9}$	1.833
$t_{0.025,\,9}$	2.262	$t_{0.05,\,15}$	1.753	$t_{0.025,\,15}$	2.131
$t_{0.05,\,24}$	1.711	$t_{0.025,\,24}$	2.064	$t_{0.05,\,29}$	1.699
$t_{0.025,\,29}$	2.045	$t_{0.05,\,40}$	1.684	$t_{0.025,\,40}$	2.021

All entries are *upper-tail* criticals. For two-sided tests use $\alpha/2$ (e.g., $z_{0.025}$ for $\alpha = 0.05$). Choose the correct sign ($\pm$) and match t df to $n - 1$.