Lecture 4: Nth Internal Interpolation Using Newton's Polynomial for Divided Differences

In this case, the interpolation function is given by:

$$y = B_0 + B_1(x - x_0) + B_2(x - x_0)(x - x_1) + ... + B_n(x - x_0)(x - x_1)...(x - x_{n-1})$$

Where we need $n+1$ to use the available data points to estimate the values of $B_0, B_1, B_2.....B_n$ By substituting all the chosen pairs into the interpolation function, using the same method that was applied in quadratic internal interpolation, we can derive the following:

$$B_0 = f[x_0] = y_0$$

$$B_1 = f[x_0, x_1] = \frac{y_{1-}\ y_0}{x_1 - x_0}$$

$$B_2 = f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{(x_2 - x_0)}$$

$$B_3 = f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{(x_3 - x_0)}$$

$$B_{n-1} = f[x_0, x_1, ..., x_{n-1}] = \frac{f[x_1, x_2 ... x_{n-1}] - f[x_0, x_1, ..., x_{n-2}]}{(x_{n-1} - x_0)}$$

$$B_n = f[x_0, x_1, ..., x_n] = \frac{f[x_1, x_2 ... x_n] - f[x_0, x_1, ..., x_{n-1}]}{(x_n - x_0)}$$

Where Where $B_n = f[x_0, x_1, ..., x_n]$ is called the *divided difference*. After calculating all the values of $B_0, B_1, B_2.....B_n$ we substitute them into the interpolation function and compute the required value.

To facilitate the calculation of the divided differences, we organize the data into a table as follows:

x_i	$y_i = f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$f[x_i, x_{i+1}, \ldots, x_{i+4}]$
x_0	$y_0 = f[x_0]$				
		$f[x_0, x_1]$			
x_1	$y_1 = f[x_1]$		$f[x_0, x_1, x_2]$		
		$f[x_1, x_2]$		$f[x_0, x_1, x_2, x_3]$	
x_2	$y_2 = f[x_2]$		$f[x_1, x_2, x_3]$		$f[x_0, x_1, x_2, \ldots, x_4]$
		$f[x_2, x_3]$		$f[x_1, x_2, x_3, x_4]$	
x_3	$y_3 = f[x_3]$		$f[x_2, x_3, x_4]$		
		$f[x_3, x_4]$			
x_4	$y_4 = f[x_4]$				

(Note that the cells highlighted in red represent the values of B_0, B_1, B_2, B_3, B_4. The values $x_0, x_1, x_2, \ldots x_4$ are displayed in the table due to page layout constraints.)

Example:

The table below shows the values of the natural logarithm of some real numbers, as given by the natural logarithmic function for each value of x :

7	6.3	6	5.5	4.5	4	x
1.945910	1.840550	1.791759	1.704748	1.504077	1.386294	y=ln(x)

Required:

Estimate the value of the natural logarithm of x = 5 using linear interpolation. Calculate the Newton's polynomial method for the dividing differences of the fourth degree (n = 4).

Solution:

The interpolation function in this case is given by:

$$y = B_0 + B_1(x - x_0) + B_2(x - x_0)(x - x_1) + B_3(x - x_0)(x - x_1)(x - x_2)$$
$$+B_4(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

Since the required model is of degree four, we need to select five data pairs to estimate the internal interpolation function. From the data table, the pairs of data we will choose are:

$$(x_0, y_0) = (4,\ 1.386294), (x_1, y_1) = (4.5,\ 1.504077), (x_2, y_2) = (5.5\ 1.704748),$$
$$(x_3, y_3) = (6,\ 1.791759), (x_4, y_4) = (6.3\ ,1.840550)$$

We will now calculate the divided differences as shown in the table below:

x_i	$y_i = f[x_i]$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	$f[x_i, x_{i+1}, ..., x_{i+4}]$
4	1.386294				
		0.235566			
4.5	1.504077		-0.023263		
		0.200671		0.002749	
5.5	1.704748		-0.017766		-0.000342
		0.174022		0.001963	
6	1.791759		-0.014232		
		0.1626367			
6.3	1.840550				

$B_0 = 1.386294 \qquad B_1 = 0.235566 \qquad B_2 = -0.023263$

$B_3 = 0.002749 \qquad B_4 = -0.000342$

Next, we substitute these values into the Newton interpolation formula.

$$y = 1.386294 + 0.235566(x-4) - 0.023263(x-4)(x-4.5)$$
$$+0.002749(x-4)(x-4.5)(x-5.5) - 0.000342\ (x-4)(x-4.5)(x-5.5)(x-6)$$

$$y = 1.386294 + 0.235566(5-4) - 0.023263(5-4)(5-4.5)$$
$$+0.002749(5-4)(5-4.5)(5-5.5) - 0.000342\ (5-4)(5-4.5)(5-5.5)(5-6)$$
$$\Rightarrow y = 1.609541$$

Thus, the estimated value of the natural logarithm of x=5 is approximately 1.609541, obtained by interpolation using Newton's polynomial method for dividing differences of the fourth degree.