Also

$$\left(\bigcap_{n=1}^{\infty} A_n\right)' = = \{0\}' = (0, \infty)$$

$$\bigcup_{n=1}^{\infty} A_n' = \bigcup_{n=1}^{\infty}\left(1 - \frac{1}{n}, \infty\right) = (0, \infty).$$

1.4. Field and σ - field

Suppose A is a class of sets. If by performing operation on one or more elements of A, we obtain an element of the same class, then we say that the class is closed under that operation. For example, if

$$A \in A \Longrightarrow A' \in A$$

then A is said to be closed under complementation. If $A, B \in A \Rightarrow A \cap B = AB \in A$., then A is said to be closed under union.

If $A, B \in A \Rightarrow A \cap B = AB \in A$, then A is said to be closed under intersection.

If $A_1, A_2, \ldots\ldots, A_n \in A$, then by induction we can show that A is closed under finite union and intersection, i.e.

$$\bigcup_{i=1}^{n} Ai \in A \text{ and } \bigcap_{i=1}^{n} Ai \in A.$$

Definition 1.4.1

A non-empty class of sets F is said to be a *field* if it is closed under complementation and finite union. Thus a field F is a non-empty class of subsets of S such that

1. If $A \in F$ and $B \in F$, then $A \cup B \in F$ (1.4.1)
2. If $A \in F$, then $A' \in F$. (1,4.2)

These two properties are sufficient to define a field. Other properties follow immediately from them.
Thus

3. If $A \in F$ and $B \in F$ then $AB \in F$ (1.4.3)
4. $S \in F$ and $\phi \in F$ (1.4.4)
5. If $A \in F$ and $B \in F$ then $A/B \in F$ (1.4.5)

Definition 1.4.2

Any field B on S is called a σ - *field* or a *Borel field* on S if it is closed under denumerable intersections and unions. Thus a Borel field B is field on S such that .

1. If $A_1, A_2, \ldots \in B$ then $\bigcup_{i=1}^{\infty} Ai \in B$

2. If $A_1, A_2, \ldots \in B$ then $\bigcap_{i=1}^{\infty} Ai \in B$

Example 1.4.1

1. Let $F = \{\phi, S\}$, then F is a field, it is the smallest field.

2. Let $A = \{A, A', \phi, S\}$, where $A \subseteq S$. Then A is a field. It is also the smallest field containing A.

3. The power set of S , i.e $\mathcal{P}(S)$ is also a field and is called the largest field .

4. let A_1, A_2 , and A_3 be three subsets of S such that $A_i \cap Aj = \phi$ for all $i,j = 1,2,3, (i \neq j)$ and $\Sigma A_i = S$. Then

$$F = \{\phi, S, A_1, A_2, A_3, A_1 \cup A_2, A_1 \cup A_3, A_2 \cup A_3\}$$

form a filed since $A_1 \cup A_2 = A_3'$, $A_1 \cup A_3 = A_2'$ and $A_2 \cup A_3 = A_1'$

The intersection of an arbitrary number of fileds is also a field but the union of two fields may not be a field, for example, let.

$F_1 = \{ \phi , S, A, A' \}$

and

$F_2 = \{ \phi , S, B, B' \}$

F_1 ant F_2 are fileds but their union $\{ \phi , S, A, A', B, B' \}$ is not a filed .

1.5 - Problems

1. Define the union and intersection of two sets.
2. Define the empty set and the space.
3. Prove equation (1.2.5) .
4. Prove equation (1.2.11) .
5. Prove equation (1.3.6) .
6. Let A and B be any two subsets of S . Prove that

a. $B/A = A' \cap B$
b. $A \cap B \subset A \subset A \cup B$
c. $A \subset B$ if and only if $A \cap B = A$.

7. Let A,B and C be any three subsets of S, Prove that

a. $A \cap (B/C) = (A \cap B)/(A \cap C)$
b. $A x (B \cap C) = (AxB) \cap (AxC)$
c. $A x (B \cup C) = (AxB) \cup ((AxC)$

8 . let $A = \{0,1,2\}$. State whether or not each statement is correct .

i . $\{0\} \in A$, (ii) $\{0,2\} \subset A$, (iii) $2 \in A$, $2 \subset A$, (iv) $\{2,1,0\} \subset A$, (v) $\phi \in A$.

9. let $A = \{w : 2w = 6\}$, $B = \{5\}$ and $C = \{x ; 3x = 9\}$. Which of these sets are equal ?

10. Let A= {1,3,4,5}, B ={3,5}, C={a,b,3,5,6} and D= {ϕ}. State whther each statement is true or false

(i) $B\subset C$, (ii) $A\subset C$, (iii) DCA ,(iv) $A\cup B=A$,
(v) $A\cap D = \phi$, (vi) $B/C = \phi$, (vii) $A/B = \{1,5\}$.

11. Let S = {1,2,3,4,5,6,7}, A={1,2,3,4,5}, B= {1,3,5,7} and C= {2,5,6,7}. Find

(i) $(A\cup C)$,(ii) $A\cap B$,(iii) C/B , (iv) $A\cap C'$,
(v) $(A/C)'$, (vi) $(A/B')'$, (vii) $(C\cap C')'$

12. Let A= {1,2,3,4}. Find

i. the power set $\mathcal{P}(A)$ of A.
ii. all the partitions of A.

13. Let S= {a,b} , W= {1,2,3,4,5,6} and V= {3,5,7,9}. Find

(i) SxW, (ii) VxW, (iii) SxWxV
(iv) $(SxW)\cap(SxV)$, (v) $Sx(W\cap V)$, (vi) $Vx(S\cup W)$

14. State whether each set is finite or infinite:

i. The set of lines parallel to the Y-axis.
ii. The set of numbers which are multiple of 3.
iii. The set of letters in the English alphabet.
iv. The set of numbers which are the solution of the polynomial $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 = 0$, n is finite.
v. The set of circles pass through the origin (0,0).
vi. The set Q of all rational numbers.
vii. The set I of all integers.
viii. The set R^+ of all positive real numbers.

15. Let A_n= {x:x is a multiple of n, n $\in$N}
= {n, 2n, 3n, ... }.

Find

(i) $A_2 \cap A_7$; (ii) $A_6 \cap A_8$; (iii). $A_3 \cup A_{12}$, (iv) $A_3 \cap A_{12}$;
(v) $A_s \cup A_{St}$, where $s,t \in N$; $(A_s \cap A_{St})$ where $s,t \in N$.

16. Prove that for any indexed class $\{A_i: i \in I\}$ and any subset B,

a. $B \cup (\bigcap_i A_i) = \bigcap_i (B \cup A_i)$
b. $B \cap (\bigcup_i A_i) = \bigcup_i (B \cap A_i)$

17. Show that each of the following is a field of subsets of S.

i. $A = \{\phi, S\}$, (ii) $B = \{\phi, A, A', S\}$, (iii) $\mathcal{P}(S)$, the power set of S.