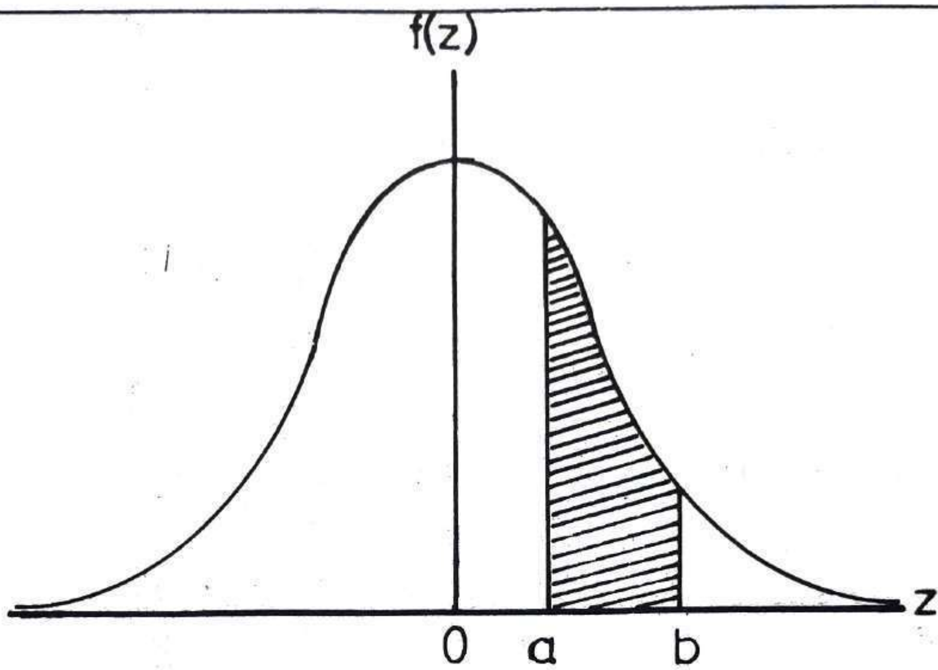

Fig 5·8

The values of ϕ (z) for negative values can be obtained from the relation

$$P(Z < -a) = P(Z > a)$$
$$= 1 - P(Z \leq a)$$
$$= 1 - \phi(a)$$
$$\therefore \phi(-a) = 1 - \phi(a).$$

For example,

1. $P(2 < Z < 3.1) = \phi(3.1) - \phi(2)$
   $= 0.999 - 0.9772 = 0.0218$

2. $P(Z \leq -0.56) = 1 - \phi(0.56)$
   $= 1 - 0.7123 = 0.2877$

If $X \sim N(\mu, \sigma^2)$, the distribution function of X can be expressed as

$F(a) = P(X \leq a)$

$$= P\left(\frac{X-\mu}{\sigma} < \frac{a-\mu}{\sigma}\right)$$

$$= P\left(Z < \frac{a-\mu}{\sigma}\right) = \phi\left(\frac{a-\mu}{\sigma}\right)$$

For example, if $X \sim N(4,25)$, then

1. $P(X < 6) = P\left(\frac{X-4}{5} < \frac{6-4}{5}\right)$

$$= P\left(Z < \frac{2}{5}\right) = \phi\ (0.4) = 0.6554$$

2. $P(3 < X < 5) = P\left(\frac{3-4}{5} < \frac{X-4}{5} < \frac{5-4}{5}\right)$

$$= P(-0.2 < Z < 0.2) = \phi\ (0.2) - \phi\ (-0.2)$$
$$= \phi\ (0.2) - [1- \phi\ (0.2)]$$
$$= 0.5793 - 1 + 0.5793 = 0.1586$$

Example 5.7.1.

Suppose the temperature in Iraq during July is normally distributed with mean 45 ^{o}C and standard derviation $\sigma = 2.5\ ^{o}C$. Find the probability that the temperature is

i. between 46.7 ^{o}C and 48.1 ^{o}C.
ii. higher than 47.5 ^{o}C.

Solution :

Let X denote the temperature during July, then $X \sim N(45, 6.25)$.

i. $P(46.7 < X < 48.1) = P\left(\frac{46 \cdot 7 - 45}{2 \cdot 5} < \frac{X - 45}{2 \cdot 5} < \frac{48 \cdot 1 - 45}{2 \cdot 5}\right)$

$= P(0.68 < Z < 1.24)$
$= \phi(1.24) - \phi(0.68) = 0.8849 - 0.7517$
$= 0.1332$

ii. $P(X > 47.5) = P\left(\frac{X - 45}{2.5} > \frac{47.5 - 45}{2.5}\right)$

$= P(Z > 1) = 1 - \phi(1)$
$= 1 - 0.8413 = 0.1587$

Example 5.7.2.

Suppose the weights of 1000 babies at birth are normally distributed with mean 8.5 pounds and standard deviation 0.5 pounds. Find the number of babies with weights

1. less than 9.7 pounds,
2. between 7.2 pounds and 8.5 pounds.

Solution:

Let X denote the weight of a baby at birth, then $X \sim N(8.5, 0.25)$.

i. $P(X < 9.7) = P\left(\frac{X - 8.5}{0.5} < \frac{9.7 - 8.5}{0.5}\right)$

$= P(Z < 2.4) = 0.9918$
Thus N= 1000 (0.9918) = 992.

ii. $P(7.2 < X < 8.5) = P\left(\frac{7.2 - 8.5}{0.5} < \frac{X - 8.5}{0.5} < \frac{8.5 - 8.5}{0.5}\right)$

$= P(-2.6 < Z < 0) = \phi(0) - \phi(-2.6)$
$= \phi(0) - [1 - \phi(2.6)]$
$= 0.5 - 1 + 0.9938 = 0.4938$
Thus N= 1000 (0.4938) = 494.

Remark:

In 1733, De-Moivre obtained the normal distribution as a limiting case of the binomial distribution and applied it to problems arising in the game of chance. Then if a r.v. X has a binomial distribution with parameters n and p, thus under the following conditions:

1. the number of trials, n, is very large,
2. neither p nor q is close to zero,

   The p. m. f. of X is approximately normal with parameters $\mu = np$ and $\sigma^2 = npq$.

Example 5.7.3.

A fair coin is tossed 20 times. Determine the probability that the number of heads are.

i. 14, (ii) between 14 and 16 inclusive, by using (a) the binomial distribution, (b) the normal approximation to the binomial distribution .

Solution

a. Let X be the number of heads occurring. In Equation (5.4.2) subsistute n=20; p= q = 1/2, we get

i. $P(X=14) = C(20,14)\left(\frac{1}{2}\right)^{14}\left(\frac{1}{2}\right)^{6}$

$$= 38760\left(\frac{1}{2}\right)^{20}$$

$$= \frac{38760}{1048576} = 0{\cdot}0369$$

ii. $P(14 \leq X \leq 16) = C(20,14)\left(\frac{1}{2}\right)^{20} + C(20,15)\left(\frac{1}{2}\right)^{20} + C(20,16)\left(\frac{1}{2}\right)^{20}$

$$= \frac{38760 + 15504 + 4845}{1048576} = \frac{59109}{104876}$$

$= 0.0563$

b. $\mu = np = 20\left(\frac{1}{2}\right) = 10$ and

$$\sigma^2 = npq = 20\ (1/2)\left(\frac{1}{2}\right) = 5$$

Since X is a continuous r.v., then

i. $P(X=14) = P(13.5 < X < 14.5)$

$$= P\left(\frac{13{\cdot}5 - 10}{\sqrt{5}} < \frac{X - 10}{\sqrt{5}} < \frac{14{\cdot}5 - 10}{\sqrt{5}}\right)$$

$$P = \left(\frac{3{\cdot}5}{2{\cdot}235} < Z < \frac{4{\cdot}5}{2{\cdot}235}\right)$$

$= P(1.57 < Z < 2.01) = \phi\ (2.01) - \phi\ (1.57)$
$= 0.9778 - 0.9418 = 0.036$

ii. $P(14 \le x \le 16) = P(13.5 \le x \le 16.5)$

$$= P\left(\frac{13{\cdot}5 - 10}{2{\cdot}235} < Z < \frac{16{\cdot}5 - 10}{2{\cdot}235}\right)$$

$= P(1.57 < Z < 2.91) = \phi\ (2.91) - \phi\ (1.57)$
$= 0.9982 - 0.9418 = 0.0564$

5.8. Problems

1. Let X be a r.v. having a p.m.f.

x	-2	-1	0	2	3	5
P(x)	c	2c	3c	2c	c	4c

1. Find the value of c.
2. Find F(x) and draw its graph.
3. Find $P(X > 2)$, $P(0 < X < 3)$, $P(X < -1)$

2. Assuming F(x) to be a distribution function of the r.v. X and is given by

$$F(x) = \begin{cases} 0 & \text{if } x < x_1 \\ 0.5 & x_1 \leq x < x_2 \\ 0.7 & x_2 \leq x < x_3 \\ 0.8 & x_3 \leq x < x_4 \\ 1 & x \geq x_4 \end{cases}$$

a. Find the p.m.f. of X.
b. Find $P(X < x_2)$, $P(X > x_4)$, $P(x_1 < X < x_3)$

3. A r.v. X has a p.d.f.

$f(x) = 3x^2 , 0 \leq x \leq 1$
$\quad\quad 0 \quad$ otherwise

Find the values of a and b such that

a. $P(X \leq a) = P(x > a)$
b. $P(X > b) = 0.1$
c. Find F(x)

4. If a r.v. X has a p.d.f.

$$f(x) = \frac{3x^2}{16} , -2 < x < 2$$
$\quad\quad 0 \quad$ otherwise

Find (i) $P(X < 1)$, (ii) $P(|X| > 1)$, (iii) $P(2X + 3 > 5)$.

5. A r.v. Y has a p.d.f.

$f(y) = ky(1-y) \quad , 0 \leq y \leq 1$

1. Find the value of k.
2. Find F(y) and draw its graph.
3. Find $P\left(\frac{1}{2} < Y < \frac{5}{4}\right)$

6. Let Z be a r.v. having a p.d.f.

$$f(z) = \begin{cases} \frac{z}{2} & 0 \le z \le 1 \\ \frac{1}{2} & 1 < z \le 2 \\ \frac{3-z}{2} & 2 < z \le 3 \end{cases}$$

1. Draw the graph of f(z).
2. Find F(z) and draw its graph.
3. Find $P(0.5 \le Z \le 2.5)$; $P(Z > 2.1)$.

7. Assume the probability that a student entering college will graduate is $\frac{2}{3}$ Find the probability that out of 5 students

a. 3 of them will graduate,
b. at most one will gradute.

8. A certain disease has a mortality rate of $\frac{1}{4}$ Seven patients suffering from this disease are selected at random, what is the probability that

i. at least 3 of them will die,
ii. all of them will die,
iii. at most 2 will recover?

9. The probability that a man misses the target is $\frac{1}{6}$.

(1) Find the probability that he hits the target 4 times if he fires 5 times. (2) Find the probability that he misses the target for the first time on his fifth shot.
(3) Find the probability that he hits the target for the third time on his tenth shot.

10. In testing a new antibiotic it was found that 2 out of 100 patients suffer an allergic reaction. Find the probability that out of 150 patients chosen at random

1. exactly 3 patients will suffer an allergic reaction,

2. more than 5 patients will suffer an allergic reaction,
3. between 2 and 4 (inclusive) patients will suffer an allergic reaction.

11. Suppose that on average 1.5 houses in every 10000 houses in a certain district have a fire during a year. If there are 30000 houses in that district, what is the probability that.
1. at least one houses will have a fire during the year?
2. exactly 4 houses will have a fire during the year?

12. The probability of getting no misprint in a page of this book is $e^{-2.5}$. What is the probability that a page contains more than 2 misprints?

13. If the probability that a person will believe a public rumor is 0.3, what is the probability that the tenth person to hear the rumor will be the fifth to believe it?

14. Patients arrive randomly and independently at a doctor's surgery from 8 A.M., at an average rate of one in ten minutes. The waiting room holds 3 persons. What is the probability that the room will be full when the doctor arrives at 9 A.M.?

15. Suppose that the length of a phone call in minutes is an exponential random variable with parameter $\theta = \frac{1}{15}$. If someone arrives immediately ahead of you at a public telephone booth, find the probability that you will have to wait (1) more than 8 minutes; (2) between 5 and 15 minutes.

16. Let X be uniformly distributed in the interval [-2,2]. Find (a) $P(X < 1)$, (b) $P\left[|X-1| \geq \frac{1}{2}\right)$

17. If the number of fish a person catches per hour at a lake is a r.v. having a Poisson distribution with m=1.8, find the probability that a person fishing there for an hour
1. will not catch any fish at all,
2. will catch at most 2.

18. In problem 17, assume that the time between two successive catches is a r.v. having an exponential distrbution with θ =1/1.8. Find the probability that it will take less than 20 minutes between successive catches.

19. Use integration by parts to show that the gamma function which is given by (5.6.4), satisfies equation (5.6.5).

20. If family annual income in a certain area (in thousand of I.D.) is a r.v. having a normal distribution with mean 1.5 and standard deviation (S.D.) of 0.16. Find the probability that a family randomly selected from this area will have

a. an annual income of less than 2000 I.D.
b. an annual income any where from 1600 I.D to 2800 I.D.
c. an annual income greater than 1850 I.D.

21. Assume the April temperature in Mosul is normally distributed with mean 20 ^{o}C and S.D. of 2 ^{o}C, find the probability that the temperature of a given day in April is

a. less than 18 ^{o}C,
b. more than 22 ^{o}C,
c. at most 21 ^{o}C.

22. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters μ =71, σ^2 =6.25. What is the percentage of 25-year old men who are over 6 feet 2 inches tall? what percentage of men are shorter than 75.5 inches?

23. Suppose the marks for an examination are normally distributed with mean 76 and S.D. 15. The top 15% of the students receive A's and the bottom 10% receive F's. Find (i) the minimum mark to receive an A and (ii) the minimum mark to pass (not to receive an F).

24. A fair die is tossed 360 times. Find the probability that the face 6 will occur (i) between 50 and 80, (inclusive) more than 75 times.