Lecture 5: Interpolation Using Lagrange Polynomial

General Formula for Interpolation Using Lagrange Polynomials (Lagrange Method):

Assume that we are given the following pairs of values$(x_0, y_0), (x_1, y_1) \dots., (x_n, y_n)$, where n is a positive integer. Suppose that we wish to build an interpolation function using the Lagrange method to estimate at y a given value of x, where
$x_0 < x < x_n \quad x_i \neq x$ for all values of i=0,1,2,….n

The general formula for the interpolation function according to the Lagrange method is:

$$y = L_0(x)y_0 + L_1(x)y_1 + \dots\dots\dots + L_n(x)y_n$$

$$= \sum_{i=0}^{n} L_i(x)y_i \dots\dots\dots\dots(1)$$

Where $L_i(x)$) is the Lagrange basis polynomial, defined as:

$$L_i(x) = \left(\frac{x - x_0}{x_i - x_0}\right)\left(\frac{x - x_1}{x_i - x_1}\right)\dots\left(\frac{x - x_{i-1}}{x_i - x_{i-1}}\right)\left(\frac{x - x_{i+1}}{x_i - x_{i+1}}\right)\dots\left(\frac{x - x_n}{x_i - x_n}\right)$$

$$= \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j} \dots\dots\dots(2)$$

The interpolation can be linear when $n = 1$, or quadratic when $n = 2$ it can be of higher degree up to n.

Linear Interpolation:

As mentioned earlier, for **linear interpolation**, we need to choose only two data points to apply the interpolation. Based on the conditions discussed in the first lecture, the two data points chosen are:

$(x_0, y_0), (x_1, y_0)$Then, using the formula for Lagrange interpolation (1) and (2), we can estimate the value of y by substituting into the following equation (where $n = 1$,in this case that is i takes the values 0 and 1 only):

$$y = \sum_{i=0}^{n=1} L_i(x) y_i$$

$$= L_0(x) y_0 + L_1(x) y_1 \text{....................(3)}$$

And $L_i(x)$be as follows when$i = 0$

$$L_0(x) = \prod_{\substack{j=0 \\ j\neq 0}}^{n=1} \frac{x - x_j}{x_0 - x_j}$$

$$= \frac{x - x_1}{x_0 - x_1}$$

And when $i = 1$:

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{n=1} \frac{x - x_j}{x_1 - x_j}$$

$$= \frac{x - x_0}{x_1 - x_0}$$

After finding the values $L_i(x)$, substitute them into the linear interpolation function (Equation 3) to calculate the value of y.

Example

From the table data below, estimate the value of y when x = 1.5 is calculated using linear interpolation, according to the Lagrange method.

x	1	1.3	1.6	1.9	2.2
y	0.1411	-0.6878	-0.9962	-0.5507	0.3115

Solution:

The chosen pairs in this case will be $(x_0, y_0) = (1.3, -0.6878)$ and $(x_1, y_1) = (1.6, -0.9962)$. We begin by finding the value of $L_i(x = 1.5)$ as follows:

$$L_0(x = 1.5) = \prod_{\substack{j=0 \\ j \neq 0}}^{n=1} \frac{x - x_j}{x_0 - x_j}$$

$$= \frac{x - x_1}{x_0 - x_1} = \frac{1.5 - 1.6}{1.3 - 1.6} = \frac{-0.1}{-0.3} = 0.3333$$

$$L_1(x = 1.5) = \prod_{\substack{j=0 \\ j \neq 1}}^{n=1} \frac{x - x_j}{x_1 - x_j}$$

$$= \frac{x - x_0}{x_1 - x_0} = \frac{1.5 - 1.3}{1.6 - 1.3} = \frac{0.2}{0.3} = 0.6667$$

The value of y can be estimated as follows:

$$y = \sum_{i=0}^{n=1} L_i\left(x = 1.5\right) y_i$$

$$= L_0\left(x = 1.5\right) y_0 + L_1\left(x = 1.5\right) y_1$$

$$= 0.3333\left(-0.6878\right) + 0.6667\left(-0.9962\right) = -0.8934$$

It should be noted that the interpolation function used above is valid for all values of x within the interval (1.6,1.3) only.

2. Quadratic Internal Interpolation

In quadratic internal interpolation, we need to select three data pairs to estimate the interpolation function. Suppose the selected pairs are (refer to the second lecture for conditions on choosing the three pairs). Then, substitute into Equations (1) and (2) above to estimate the value of y at a specific value of x, as follows:

$$y = \sum_{i=0}^{n=2} L_i\left(x\right) y_i$$

$$= L_0\left(x\right) y_0 + L_1\left(x\right) y_1 + L_2\left(x\right) y_2 \text{.....(4)}$$

calculated as follows when:

$$L_0(x)=\prod_{\substack{j=0\\ j\neq 0}}^{n=2}\frac{x-x_j}{x_0-x_j}$$

$$=\left(\frac{x-x_1}{x_0-x_1}\right)\left(\frac{x-x_2}{x_0-x_2}\right)$$

when $i=1$:

$$L_1(x)=\prod_{j=0}^{n=2}\frac{x-x_j}{x_1-x_j}$$

$$j\neq 1$$

$$=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_2}{x_1-x_2}\right)$$

And when $i=2$:

$$L_2(x)=\prod_{\substack{j=0\\ j\neq 2}}^{n=2}\frac{x-x_j}{x_2-x_j}$$

$$=\left(\frac{x-x_0}{x_2-x_0}\right)\left(\frac{x-x_1}{x_2-x_1}\right)$$

After finding the $L_i(x)$ values, we substitute them into the quadratic interpolation function (Equation 4) to calculate the value of y.

Example:
For the data from the previous example, estimate the value of y when $x = 1.5$ using quadratic interpolation via the Lagrange polynomial.

Solution:
As mentioned earlier, we need to select three pairs of data to estimate the quadratic model. The selected pairs in this case are:

$$(x_0, y_0) = (1.3, -0.6878), (x_1, y_1) = (1.6, -0.9962),$$
$$(x_2, y_2) = (1.9, -0.5507)$$

We begin by calculating the value of $L_i(x = 1.5)$ as follows:

$$L_0(x = 1.5) = \prod_{\substack{j=0 \\ j \neq 0}}^{n=2} \frac{x - x_j}{x_0 - x_j}$$

$$= \left(\frac{x - x_1}{x_0 - x_1}\right)\left(\frac{x - x_2}{x_0 - x_2}\right) = \left(\frac{1.5 - 1.6}{1.3 - 1.6}\right)\left(\frac{1.5 - 1.9}{1.3 - 1.9}\right)$$
$$= \left(\frac{-0.1}{-0.3}\right)\left(\frac{-0.4}{-0.6}\right) = 0.2222$$

$$L_1(x) = \prod_{\substack{j=0 \\ j\neq 1}}^{n=2} \frac{x - x_j}{x_1 - x_j}$$

$$= \left(\frac{x - x_0}{x_1 - x_0}\right)\left(\frac{x - x_2}{x_1 - x_2}\right) = \left(\frac{1.5 - 1.3}{1.6 - 1.3}\right)\left(\frac{1.5 - 1.9}{1.6 - 1.9}\right)$$

$$= \left(\frac{0.2}{0.3}\right)\left(\frac{-0.4}{-0.3}\right) = 0.8889$$

$$L_2(x) = \prod_{\substack{j=0 \\ j\neq 2}}^{n=2} \frac{x - x_j}{x_2 - x_j}$$

$$= \left(\frac{x - x_0}{x_2 - x_0}\right)\left(\frac{x - x_1}{x_2 - x_1}\right) = \left(\frac{1.5 - 1.3}{1.9 - 1.3}\right)\left(\frac{1.5 - 1.6}{1.9 - 1.6}\right)$$

$$= \left(\frac{0.2}{0.6}\right)\left(\frac{-0.1}{0.3}\right) = -0.1111$$

Then we can estimate the value of y as follows:

$$
\begin{aligned}
y &= \sum_{i=0}^{n=2} L_i\left(x=1.5\right) y_i \\
&= L_0\left(x=1.5\right) y_0 + L_1\left(x=1.5\right) y_1 + L_2\left(x=1.5\right) y_2 \\
&= 0.222(-0.6878) + 0.8889(-0.9962) - 0.1111(-0.5507) \\
&= -0.9772
\end{aligned}
$$

Note that the interpolation function used above is valid for all values of x within the interval $(1.3\ ,1.9)$only.