2

Chapter Two

Techniques of Counting

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In this chapter we give some techniques which are not of much use in defining a numerical measure for chances of occurrence of various events, but will be of help in evaluating probabilities of a certain event which will be discussed in the following chapters. Sometimes these techniques are referred to as combinatorial analysis.

2.1 Fundamental Principle of Counting

If one thing can happen in n_1 different ways, and after this a second thing can happen in n_2 different ways, ... and finally a k th thing can happen in n_k different ways, then all k things can happen in the specified order in $n_1 n_2 \ldots n_k$ different ways:

Example 2.1.1

1. A drug store stocks toothpaste from seven different manufactures. Each manufacture puts out three sizes, each available in fluoridated form and plain. Then the drug store stocks 7. 3. 2= 42 different kinds of toothpaste tubes.

2. A man decides to go to Europe by plane and to return by ship. If there are 8 different airlines available to him, and nine different shipping companies, then he can make his round trip in 8.9 = 72 different ways.

Example 2.1.2.

Suppose a car number plate contains three distinct letters followed by two digits with the first digit not a zero.

Since there are 26 letters in the English alphabet, then the first letter can be printed in 26 different ways, the 2nd letter can be printed in 25 different ways and the 3rd letter can be printed in 24 letters.

Now, since there are 10 digits which are 0, 1, 2, ..., 9, therefore the first digit can be printed in 9 different ways and the second digit can be printed in 10 different ways. Hence we can print.

26. 25. 24. 9.10 = 1404000

different car number plates.

2.2. Permutations

Let us consider the set of 3-letter words that can be form -ed from A, M, and N.

These are

AMN MAN NAM ANM MNA NMA

Thus there are 6 different arrangements of the letters A, M, and N. Each arrangement is called a permutation, so we give the following definition

Defintion 2.2.1.

An arrangement of a set of n different objects in a given order is called a *permutation* of the objects (taken all at a time).

The arrangement of any $r \leq n$ of these objects in a given order is called *a permutation* of n objects taken r at a time, and is denoted by $^n p_r$ or P(n, r).

Before we give the general formula for P(n,r), let us define the *factorial notation.* It is the product of the positive integers from 1 to n inclusive and is denoted by n! (read n factorial), i.e.

$$n! = n(n-1) \ldots\ldots 2.1 \qquad (2.2.1)$$

with $o! = 1$ and $1! = 1$

Also we can write $n!$ as

$$n! = n(n-1)!, \; n \geq 1 \qquad (2.2.2)$$

For example,

$5! = 5.4.3.21 = 120,$

$8! = 8.7.6.5.4.3.2.1 = 23040$

and $\dfrac{8!}{5!} = 8.7.6 = 336.$

1. The number of *permutuations* of n different objects taken *all at a time* is n!.

2. The number of *permutations* of n different objects taken r *at a time* is P(n,r), and is defined as

$$P(n,r) = n(n-1)\,(n-2) \ldots\ldots (n-r+1)$$

$$= \frac{n!}{(n-r)!} \quad , 0 \leq r \leq n. \qquad (2.2.3)$$

For example, $P(6,4) = \dfrac{6!}{2!} = 360$, $P(5,5) = \dfrac{5!}{0!} = 120.$

3. The total number of arrangements of n different objects in r places $(r \leq n)$ is equal to n^r, if repetition is allowed.

The permutations that we have considered so far are called linear permutations because they are permutations of objects in a line or in a row. Permutations of objects in a circle are called circular permutations, so we have

4. The total no. of *arrangements* of n different objects *around a circle* is (n-1)!

5. The number of arrangements of n objects such that r_1 of them are of one kind, r_2 of them of a second kind, ... , r_k of the k-th kind, is denoted by $nP_{r_1, r_2, \ldots, r_k}$ and is given by

$$nP_{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1!\, r_2! \ldots r_k!}, \; r_1 + r_2 + \ldots + r_k = n. \quad \ldots(2{\cdot}2{\cdot}4)$$

Example 2.2.1

How many 3-digit numbers can be formed from the digits 2,3,4,5,6,7 if (a) repetitions are not allowed, (b) repetitions are allowed.

a. If repetitions are not allowed, then the first digit can be chosen in 6 ways, the 2nd digit in 5 ways and the 3rd in 4 ways, so the total number that can be formed is 6.5.4=120.

Another method

Since repetitions are not allowed, then the total number that can be formed is

$$P(6,3) = \frac{6!}{3!} = 6.5.4 = 120.$$

b. If repetitions are allowed, then each digit can be. chosen in 6 ways, so the total number is $6^3 = 216$.

Example 2.2.2.

In how many ways can 7 persons be seated

i. on a bench?
ii. around a circular table ?

Solution

i. The seven persons can be seated on a bench in 7!=5040 ways.

ii. One person can sit at any place at the circular table. The other six person can then arrange themselves in 6! =720 ways around the table.

Example 2.2.3

1. The number of different permutations of the letters of the word Statistics, which consists of 3 s's, 3 t's, 2 i's, 1 a and 1 c, is

$$10P_{3,3,2,1,1} = \frac{10!}{3!\,3!\,2!} = 50400.$$

2. The number of different signals, each consisting of 7 flags hung in a vertical line, can be formed from 4 identical red flags, 2 identical white and one blue, is

$$7P_{4,2,1} = \frac{7!}{4!\,2!\,1!} = 105.$$

2.3. Combinations and Binomial Expansion

2.3.1 Combinations

In permutations we are interested in the order of arrangement of the objects. If one is intersted only in what particular objects are selected when r objects are chosen from n objects, without regard to their arrangement in a line, then the unordered r arrangement is called a combination. For example, the combinations of the letters a,b,c,d taken 3 at a time are: abc abd acd bcd

Definition 2.3.1

The number of *combinations* of r objects selected from a set of n objects, denoted C(n,r), (or $\binom{n}{r}$, $^{n}C_{r}$) is defined as