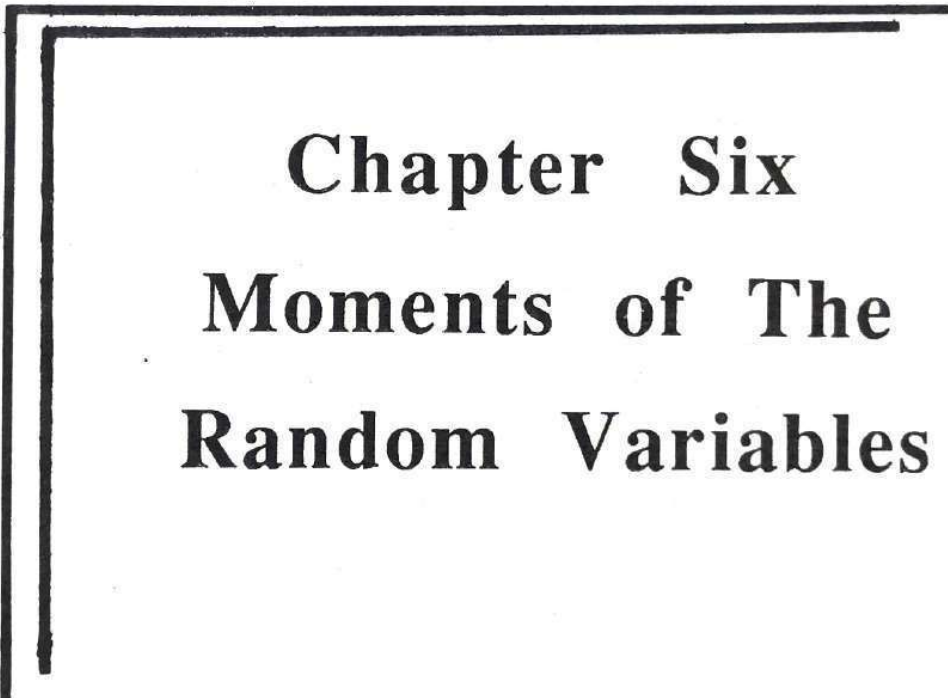


# 6



**Chapter Six**  
**Moments of The**  
**Random Variables**

## Chapter Six

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### Moments of The Random Variables

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#### 6.1. Expectation of a Discrete Random Variable

The reader may be familiar with the integration and differentiation operators in Calculus. Here we define an operator called "Expectation" which is one of the most important concepts in probability theory.

**Definition 6.1.1.**

The *expectation* of a discrete random variable  $X$  having a p.m.f.  $P(x)$ , is denoted by  $E(X)$ , and is defined by

$$E(x) = \sum_i x_i P(x_i) \quad \dots\dots\dots (6.1.1)$$

whenever the sum is absolutely convergent. In other words, the expected value of  $X$  is a weighted average of the possible values that  $X$  can take on, each value  $x_i$  being weighted by the probability  $P(x_i)$  that  $X$  assumes, for  $i=1,2,\dots$ . We refer to  $E(x)$  as the mean value of  $X$ ; it is the position of the center of gravity.

**Example 6.1.1.**

1. Suppose you get 2.00 I.D. if a head occurs and 3.00 I.D. if a tail occurs when a fair coin is tossed, then the total amount of money you get is

$$\begin{aligned} E(X) &= \sum x_i p(x_i) \\ &= 2 \left( \frac{1}{2} \right) + 3 \left( \frac{1}{2} \right) = 2.5 \text{ I. D.} \end{aligned}$$

2. Assume you get 4 I.D. if 1,2 or 5 appears, and lose 2 I.D. if 3,4 or 6 appears when a fair die is tossed once, then the total amount of money you get is

$$E(X) = \sum x_i P(x_i)$$

where

$x_1, x_2$  and  $x_5$  are equal to 4, while  $x_3, x_4$  and  $x_6$  are equal to -2, then

$$E(X) = 4 \left( \frac{1}{6} \right) + 4 \left( \frac{1}{6} \right) + (-2) \left( \frac{1}{6} \right) + (-2) \left( \frac{1}{6} \right) + 4 \left( \frac{1}{6} \right) + (-2) \left( \frac{1}{6} \right) = 2 - 1 = 1 \text{ I.D.}$$

### Example 6.1.2.

Let  $X$  denote the number of boys in a family of 4 children, then the expected number of boys is

$$E(X) = \sum_{x=0}^4 x \cdot P(x)$$

where  $P(x) = C(4,x) \left( \frac{1}{2} \right)^x \left( \frac{1}{2} \right)^{4-x}$ ,  $x=0,1,2,3$  and 4.

Then, the p.m.f. of  $X$  is

$x$	0	1	2	3	4
$P(x)$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$	$\frac{1}{16}$

Therefore,

$$\begin{aligned}
 E(X) &= \sum x p(x) \\
 &= 0 \left( \frac{1}{16} \right) + 1 \left( \frac{4}{16} \right) + 2 \left( \frac{6}{16} \right) + 3 \left( \frac{4}{16} \right) + 4 \left( \frac{1}{16} \right) \\
 &= 2
 \end{aligned}$$

**Example 6.1.3.**

In a lottery there are 100 prizes of 10 I.D., 60 prizes of 25 I.D., 40 prizes of 50 I.D. and 10 prizes of 100 I.D. Assuming that 5000 tickets are to be issued and sold, what is a fair price to pay for the ticket?

**Solution:**

Let  $X$  denote the amount of money to be won on a ticket. Then the p.m.f. of  $X$  is

$x_i$	0	10	25	50	100
$P(x_i)$	$\frac{479}{500}$	$\frac{10}{500}$	$\frac{6}{500}$	$\frac{4}{500}$	$\frac{1}{500}$

Therefore

$$\begin{aligned}
 E(X) &= \sum_{i=1}^5 x_i P(x_i) \\
 &= 0 \left( \frac{479}{500} \right) + 10 \left( \frac{10}{500} \right) + 25 \left( \frac{6}{500} \right) + 50 \left( \frac{4}{500} \right) + 100 \left( \frac{1}{500} \right) \\
 &= \frac{550}{500} = 1.1 \text{ I.D.}
 \end{aligned}$$

Thus the fair price to pay for one ticket is 1.1 I.D.

## 6.2 Expectation of Some Discrete Random Variables

1. Let  $X$  be a *Bernoulli* r.v. with parameter  $p$  whose p.m.f. is given by (5.4.1), then

$$E(X) = 0(1-p) + 1(p) = p$$

2. Let  $X$  be a binomial r.v. with parameters  $n$  and  $p$  whose p.m.f. is given by (5.4.2), then

$$\begin{aligned} E(X) &= \sum_{x=0}^n x \cdot P(x) = \sum_{x=0}^n x \cdot C(n,x) p^x q^{n-x} \\ &= \sum_{x=1}^n x \cdot \frac{n!}{x!(n-x)!} p^x q^{n-x} \\ &= \sum_{x=1}^n \frac{x \cdot n(n-1)!}{x(x-1)!(n-x)!} p^x q^{n-x} \\ &= np \sum_{y=0}^{n-1} \frac{n!}{y!(n-y)!} p^y q^{n-y}, \text{ where} \end{aligned}$$

$y = x-1$  and  $m = n-1$ , then by binomial theorem  
 $E(X) = np(p+q)^m = np$

Thus the expected value of the binomial distribution is  $np$ .

3. Let  $X$  be a *Poisson* r.v. with parameter  $m$  whose p.m.f. is given by (5.4.3), then

$$\begin{aligned} E(X) &= \sum_{x=0}^{\infty} x P(x) = \sum_{x=1}^{\infty} \frac{x e^{-m} m^x}{x!} \\ &= \sum_{x=1}^{\infty} \frac{x e^{-m} m^x}{x(x-1)!} = m \sum_{x=1}^{\infty} \frac{e^{-m} m^{x-1}}{(x-1)!} \\ &= m \sum_{y=0}^{\infty} \frac{e^{-m} m^y}{y!}, \text{ where } y = x-1 \end{aligned}$$

then

$$E(X) = m \cdot 1 = m.$$

Thus the expected value of the Poisson distribution is  $m$ .

4. Let  $X$  be a r.v. with a *geometric* distribution whose p.m.f. is given by (5.4.4.) , then

$$\begin{aligned} E(X) &= \sum_{x=1}^{\infty} x P(x) \\ &= \sum_{x=1}^{\infty} x \cdot pq^{x-1} = p \sum_{x=1}^{\infty} x q^{x-1} \end{aligned}$$

We have  $\sum_{x=0}^{\infty} q^x = \frac{1}{1-q}$ , then

$$\begin{aligned} \frac{d}{dq} \left( \sum_{x=0}^{\infty} q^x \right) &= \sum_{x=1}^{\infty} x q^{x-1} = \frac{d}{dq} \left( \frac{1}{1-q} \right) \\ &= \frac{1}{(1-q)^2} \end{aligned}$$

Therefore

$$E(X) = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

Thus the expected value of geometric distribution with

parameter  $p$  is  $\frac{1}{p}$

### 6.3. Expectation of a Continuous Random Variable

#### Definition 6.3.1

The *expectation* of a continuous random variable with density function  $f(x)$  is

$$E(X) = \int_{-\infty}^{\infty} x f(x) dx \quad \dots (6.3.1)$$

wherever this integral exists.

#### Example 6.3.1.

1. Let  $X$  be a r.v. having a p.d.f. given by

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

then

$$\begin{aligned} E(X) &= \int_{-1}^2 x \frac{x^2}{3} dx = \left[ \frac{x^4}{12} \right]_{-1}^2 \\ &= \frac{16}{12} - \frac{1}{12} = \frac{5}{4} \end{aligned}$$

2. Let  $X$  be a r.v. with a p.d.f. given by

$$f(x) = \begin{cases} \frac{1}{2}x & , 0 \leq x \leq 1 \\ \frac{1}{2} & , 1 < x < 2 \\ \frac{3}{2} - \frac{1}{2}x & , 2 \leq x \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

then

$$\begin{aligned} E(X) &= \int_{-\infty}^{\infty} x f(x) dx \\ &= \int_0^1 x \cdot \frac{1}{2} x dx + \int_1^2 x \cdot \frac{1}{2} dx + \int_2^3 x \left( \frac{3}{2} \right. \\ &\quad \left. - \frac{1}{2} x \right) dx \\ &= \left[ \frac{x^3}{6} \right]_0^1 + \left[ \frac{x^2}{4} \right]_1^2 + \left[ \frac{3x^2}{4} - \frac{x^3}{6} \right]_2^3 \\ &= \frac{1}{6} + \frac{3}{4} + \frac{7}{12} = \frac{18}{12} = \frac{3}{2} . \end{aligned}$$

#### 6.4. Expectation of Some Continuous Random Variables

1. Let  $X$  be a r.v. with a *uniform distribution* (5.6.1) in the interval  $[a, b]$  then

$$\begin{aligned} E(X) &= \int_a^b x \cdot \frac{1}{b-a} dx = \left[ \frac{x^2}{2(b-a)} \right]_a^b \\ &= \frac{b^2 - a^2}{2(b-a)} = \frac{b+a}{2} \end{aligned}$$

2. Let  $X$  be a r.v. having an *exponential distribution* (5.6.2) with parameter  $1/\theta$  then

$$\begin{aligned}
E(X) &= \int_0^{\infty} x \frac{1}{\theta} e^{-x/\theta} dx \text{ (Integrating by parts)} \\
&= -xe^{-x/\theta} \Big|_0^{\infty} + \int_0^{\infty} e^{-x/\theta} dx \\
&= 0 - \theta e^{-x/\theta} \Big|_0^{\infty} \\
&= -[0 - \theta] = \theta
\end{aligned}$$

3. Let  $X$  be a r.v. having a *normal distribution* (5.7.1) with parameters  $\mu$  and  $\sigma^2$ , then

$$\begin{aligned}
E(X) &= \int_{-\infty}^{\infty} x f(x) dx = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx \\
&= \int_{-\infty}^{\infty} (x-\mu) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx \\
&\quad + \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx
\end{aligned}$$

By symmetry, the first integral is equal to zero, then  $E(X) = \mu \cdot 1 = \mu$ .

## 6.5. Function of a Random Variable and Its Expectation

If  $X$  is a r.v. and  $g$  is any function, then  $Y = g(X)$  is also a r.v. Suppose we want to obtain the probability distribution for  $Y$ , given the probability distribution of a r.v.  $X$  and a relationship of the form  $y=g(x)$  between the values of  $X$  and those of  $Y$ , In the discrete case there is no

problem, for all we have to do is to make the appropriate substitutions. For example, suppose that  $X$  is a r.v. with the p.m.f.

$X = x$	- 2	- 1	0	1	2	3
$P(x)$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{1}{9}$

If we let  $Y = X^2$ , it follows that the values of  $Y$  are 0,1,4 and 9 with the probabilities

$$h(y=0) = P(x=0) = \frac{1}{9},$$

$$h(y=1) = P(x=-1 \text{ or } 1) = \frac{4}{9}$$

$$h(y=4) = P(x=-2 \text{ or } x=2) = \frac{3}{9}$$

and  $h(y=9) = P(x=3) = \frac{1}{9}$

Hence we can find the expectation of  $Y$  as follows:

$$E(Y) = \sum y h(y)$$

$$= 0 \left( \frac{1}{9} \right) + 1 \left( \frac{4}{9} \right) + 4 \left( \frac{3}{9} \right) + 9 \left( \frac{1}{9} \right)$$

$$= \frac{4 + 12 + 9}{9} = \frac{25}{9}$$

In the continuous case, we shall assume that the function given by  $Y=g(X)$  is differentiable and either