

Lecture 6: Numerical Differentiation

Traditional Definition of Derivative:

Let $f(x)$ be a continuous and differentiable function at a point x . Then, the derivative $f'(x)$ (denoted as $\frac{df(x)}{dx}$ or $f'(x)$) is defined as the ratio of the change in the value of the function $f(x)$ to the change in the value x of the variable, which is mathematically expressed as:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \dots\dots\dots(1)$$

Example:

For $f(x) = 5x^2$, and from our previous knowledge of the laws of differentiation, the derivative of this function is:

$$f'(x) = 5(2x) = 10x$$

Now, using the definition above (Equation 1), we can find the derivative of the function:

$$\begin{aligned} f'(x) &= \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{5(x + \Delta x)^2 - 5(x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{5(x^2 + (\Delta x)^2 + 2x\Delta x) - 5(x)^2}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{5x^2 + 5(\Delta x)^2 + 10x\Delta x - 5(x)^2}{\Delta x} \\ &= \lim_{\Delta x \rightarrow 0} \frac{5(\Delta x)^2 + 10x\Delta x}{\Delta x} = \lim_{\Delta x \rightarrow 0} 5\Delta x + 10x = 0 + 10x = 10x \end{aligned}$$

This is the same result we obtained using the differentiation rules.

In some cases, finding the derivative using the standard methods may be difficult or impossible, so we resort to numerical methods. In the following sections, we will discuss some methods for numerically estimating derivatives.

1. Forward Divided Differences (FDD):

Let $f(x)$ be a continuous and differentiable function at a point x .

Then, the derivative $f'(x)$ can be approximated as follows:

$$f'(x) \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

This is the same definition of the derivative, but without taking the limit (i.e., without $\lim_{\Delta x \rightarrow 0}$). The last fraction is called the forward divided difference (FDD).

Example:

Estimate the value of the derivative of the following function using the forward divided difference method when $x = 3$ and $\Delta x = 0.1$.

$$f(x) = 2e^{1.5x}$$

Solution:

$$\begin{aligned}f'(x=3) &= \frac{f(x+\Delta x) - f(x)}{\Delta x} \\ &= \frac{f(3+0.1) - f(3)}{0.1} \\ &= \frac{f(3.1) - f(3)}{0.1} \\ &= \frac{2e^{1.5 \times 3.1} - 2e^{1.5 \times 3}}{0.1} = 291.3571\end{aligned}$$

The last value is the true value of the derivative when when $x = 3$
Thus, the absolute error in the estimation is:

$$\text{Absolute Error} = |270.0514 - 291.3571| = 21.3057$$

Note that the derivative estimate is affected by the value of Δx , and thus the errors also change. When Δx changes, the estimated value of the derivative changes. The following table shows the differences in the estimated value of the derivative and the absolute error as Δx changes.

Δx	$f'(x)$ (FDD)	Absolute error
0.1	291.3571	21.3057
0.05	280.4363	10.3849
0.025	275.1787	5.1273
0.01	272.0869	2.0355

2. Backward Divided Differences (BDD):

Let $f(x)$ be a continuous and differentiable function at a point x .

Then, the derivative $f'(x)$ can be estimated as follows:

$$f'(x) \approx \frac{f(x) - f(x - \Delta x)}{\Delta x}$$

The last fraction is called the backward divided difference (BDD).

Example:

Estimate the value of the derivative of the following function using the backward divided difference method when $x = 3$ and $\Delta x = 0.1$.

$$f(x) = 2e^{1.5x}$$

Solution:

The last value is the estimated derivative of the function when $x = 3$ and $\Delta x = 0.1$ using the backward divided difference method.

$$\begin{aligned}
f'(x=3) &\approx \frac{f(x) - f(x - \Delta x)}{\Delta x} \\
&= \frac{f(3) - f(3 - 0.1)}{0.1} \\
&= \frac{f(3) - f(2.9)}{0.1} \\
&= \frac{2e^{1.5 \times 3} - 2e^{1.5 \times 2.9}}{0.1} = 250.7734
\end{aligned}$$

The true derivative value when $x = 3$ is:

$$f'(x=3) = 270.0514$$

Thus, the absolute error in the estimation is:

$$\text{Absolute Error} = |270.0514 - 250.7734| = 19.278$$

As previously noted, the accuracy of the estimate depends on the value of Δx . The following table shows the differences in the estimated value of the derivative and the absolute and relative errors as Δx changes.

Δx	$f'(x)$ (BDD)	Absolute error
0.1	250.7734	19.2780
0.05	260.173	9.8784
0.025	265.0506	5.0008
0.01	268.0361	2.0153

3. Central Divided Differences (CDD):

Let $f(x)$ be a continuous and differentiable function at a point x .

Then, the derivative $f'(x)$ can be estimated as follows:

$$f'(x) \approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x}$$

The last fraction is called the central divided difference (CDD).

Example:

Estimate the value of the derivative of the following function using the central divided difference method when $x=3$, $\Delta x = 0.1$.

Solution:

$$f(x) = 2e^{1.5x}$$

$$\begin{aligned} f'(x=3) &\approx \frac{f(x + \Delta x) - f(x - \Delta x)}{2\Delta x} \\ &= \frac{f(3 + 0.1) - f(3 - 0.1)}{2 \times 0.1} \\ &= \frac{f(3.1) - f(2.9)}{0.2} \\ &= \frac{2e^{1.5 \times 3} - 2e^{1.5 \times 2.9}}{0.2} = 271.0652 \end{aligned}$$

Thus, the absolute error in the estimation is:

$$\text{Absolute Error} = |270.0514 - 271.0652| = 1.0138$$

It is clear that the error values in this method are much smaller compared to the errors in the previous methods at the same $\Delta x = 0.1$. The table below shows the absolute and relative errors for different values of Δx .

Δx	$f'(x)$ (CDD)	Absolute error
0.1	271.0652	1.0138
0.05	270.3046	0.2532
0.025	270.1147	0.0633
0.01	270.0615	0.0101

Homework:

Calculate the value of the derivative of the function below using the three methods described above $f(x) = \sqrt{x}$ at the point $x = 3$,

$\Delta x = 0.1$ then calculate the absolute error for each method, knowing

that the real derivative is: $f'(x) = \frac{1}{2\sqrt{x}}$