

- ii. One person can sit at any place at the circular table. The other six person can then arrange themselves in $6! = 720$ ways around the table.

Example 2.2.3

1. The number of different permutations of the letters of the word Statistics, which consists of 3 s's, 3 t's, 2 i's, 1 a and 1 c, is

$${}^{10}P_{3,3,2,1,1} = \frac{10!}{3!3!2!} = 50400.$$

2. The number of different signals, each consisting of 7 flags hung in a vertical line, can be formed from 4 identical red flags, 2 identical white and one blue, is

$${}^7P_{4,2,1} = \frac{7!}{4!2!1!} = 105.$$

2.3. Combinations and Binomial Expansion

2.3.1 Combinations

In permutations we are interested in the order of arrangement of the objects. If one is interested only in what particular objects are selected when r objects are chosen from n objects, without regard to their arrangement in a line, then the unordered r arrangement is called a combination. For example, the combinations of the letters a,b,c,d taken 3 at a time are: abc abd acd bcd

Definition 2.3.1

The number of *combinations* of r objects selected from a set of n objects, denoted $C(n,r)$, (or $\binom{n}{r}$, nC_r) is defined as

$$C(n, r) = \frac{n(n-1)\dots(n-r+1)}{r!} = \frac{P(n, r)}{r!}$$

$$= \frac{n!}{r!(n-r)!}, 0 \leq r \leq n. \quad \dots (2.3.1)$$

Formula (2.3.1) holds for any positive integer n and $r=0,1,2, \dots, n$.

For example : $C(5,1) = \frac{5!}{1!4!} = 5$, $C(6,2) = \frac{6!}{2!4!} = 15$.

Example 2.3.1

1. How many committees of 4 persons can be formed out of 10 persons?

Solution : The total number of committees are

$$C(10,4) = \frac{10!}{4!6!} = 210.$$

2. In how many ways can a student answer 6 questions out of 8 questions in an exam?

Solution : The 6 questions can be selected in $C(8,6) = \frac{8!}{6!2!} = 28$ ways.

3. Find the number of subsets of a set A containing 5 elements.

Solution : The number of subsets of A with r elements ($r \leq 5$) is given by $C(5,r)$ for $r=0,1,2, \dots, 5$.

Hence the total number is

$$C(5,0) + C(5,1) + C(5,2) + C(5,3) + C(5,4) + C(5,5)$$

$$= 1 + 5 + 10 + 10 + 5 + 1 = 32.$$

2.3.2 Binomial Expansion

It is customary to refer to the symbol $C(n,r)$ as a *binomial coefficient*.

To state and prove some theorems related to binomial coefficients, let us make the definition that $C(n,r) = 0$ whenever n is a positive integer and r is a positive integer greater than n .

For any positive integers n and $r=0,1,2,\dots,n$, it is evident from formula (2.3.1) that

$$C(n,r) = C(n,n-r).$$

Theorem 2.3.1.

$$C(n,r) = C(n-1,r) + C(n-1,r-1) \quad (2.3.2)$$

for $r = 1, 2, \dots, n-1$.

Proof:

By using formulas (2.3.1) and (2.2.2), the R.H.S. of (2.3.2) is

$$\begin{aligned} C(n-1,r) + C(n-1,r-1) &= \frac{(n-1)!}{r!(n-1-r)!} + \frac{(n-1)!}{(r-1)!(n-r)!} \\ &= \frac{(n-1)!}{(r-1)!(n-r-1)!} \left[\frac{1}{r} + \frac{1}{n-r} \right] \\ &= \frac{n(n-1)!}{r(r-1)!(n-r-1)!(n-r)} \\ &= \frac{n!}{r!(n-r)!} = C(n,r) \end{aligned}$$

Herein, we give a formula which is useful to approximate $n!$, where n is large. It is called *Stirling's formula* :

$$n! \simeq \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$$

The symbol \simeq means approximately equal to. Where e is the base of natural logarithm, $e \simeq 2.7183$. For example,

$$\begin{aligned} 10! &\simeq \sqrt{2 \left(\frac{22}{7} \right) (10)} \left(\frac{10}{2.7183} \right)^{10} \\ &\simeq 7.9282 (3.67876)^{10} \\ &= 3599060.5 \end{aligned}$$

Theorem 2.3.2 - (Binomial Theorem).

$$(a + b)^n = \sum_{r=0}^n C(n, r) \cdot a^r b^{n-r} \quad \dots (2.3.3)$$

Proof

We shall prove the theorem by induction. The theorem is true for $n=1$ and 2 . Assume it is true for $n=k$, i.e.

$$(a + b)^k = \sum_{r=0}^k C(k, r) a^r b^{k-r}$$

and we shall show that it is true for $n=k+1$. Thus

$$\begin{aligned} (a + b)^{k+1} &= (a + b)(a + b)^k \\ &= (a + b) \left[\sum_{r=0}^k C(k, r) a^r b^{k-r} \right] \end{aligned}$$