increasing or decreasing for all values within the range of x which $f(x) \neq 0$, then the probability density function of Y is given by

$$h(y) = f(x) . \left| \frac{dx}{dy} \right| , \quad \frac{dy}{dx} \neq 0 \qquad \ldots (6 \cdot 5 \cdot 1)$$

For example, if X is uniformly distributed over (0,2) and if $Y = e^X$, then $x = \ln y$,

$$\frac{dx}{dy} = \frac{1}{y}, y \neq 0$$

We have $f(x) = \frac{1}{2}$, $0 < x < 2$
0 otherwise

Therefore, by (6.5.1) the density function of Y is

$h(y) = \frac{1}{2} \cdot \frac{1}{y}$ for $1 < y < e^2$, and the range of y is obtained as follows

For x=o, $y=e^0 = 1$ and for x= 2, $y= e^2$.

Hence we can find the expectation of Y as follows

$$E(Y) = \int_1^{e^2} y \frac{1}{2y} dy = \frac{1}{2} y \Big]_1^{e^2} = \frac{e^2 - 1}{2}.$$

Proposition 6.5.1.

1. If X is a discrete r.v. with p.m.f. P(x), then for any real-valued function $Y = g(X)$,

$$E(Y) = E[g(X)] = \sum_x g(x) P(x) \qquad \ldots (6 \cdot 5 \cdot 2)$$

2. If X is a continuous r.v. with p,d.f. f(x), then for any real-valued function Y= g(X),

$$E(Y) = E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx \qquad \dots (6\cdot5\cdot3)$$

Example 6.5.1.

If a r. v. X has the geometric distribution

$$P(x) = p(1-p)^{x-1}, x = 1, 2, \dots$$

and the values of the r.v. Y are related to those of X by means of the equation y= 3- 2x, then the probability distribution of Y is

$h(y) = f(x)$,

where $x = \dfrac{3-y}{2}$.

hence

$$h(y) = p(1-p)^{\frac{3-y}{2}-1} = p(1-p)^{\frac{1-y}{2}} \quad \text{for } y= 1, -1, -3, -5, \dots..$$

Thus, by applying (6.5.2), we get

$$E(Y) = \Sigma g(x) P(x)$$

$$= \sum_{x=1}^{\infty} (3-2x) p(1-p)^{x-1}$$

$$= 3 \sum_{x=1}^{\infty} p(1-p)^{x-q} - 2 \sum_{x=1}^{\infty} x\, p(1-p)^{x-1}$$

$= 3 - 2/p$

This result can also be obtained if we apply the formula

$$E(Y) = \Sigma\, yh(y) \qquad \text{for } y = 1, -1, -3, -5, \ldots$$

$$= \Sigma\, y\, p(1-p)^{\frac{1-y}{2}}$$

$$p + (-1)pq + (-3)pq^2 + (-5)pq^3 + \ldots$$

$$= p - pq[1 + 3q + 5q^2 + 7q^3 + \ldots]$$

$$= p - pq\left[\sum_{n=0}^{\infty} (2n+1)q^n\right]$$

$$= p - pq\left[2\sum_{n=0}^{\infty} n\,q^n + \sum_{n=0}^{\infty} q^n\right]$$

$$= p - 2pq^2 \sum_{n=0}^{\infty} n\,q^{n-1} - pq\,\Sigma\, q^n$$

$$= p - 2pq^2 \cdot \frac{1}{(1-q)^2} - \frac{pq}{1-q}$$

$$= p - \frac{2q^2}{p} + q$$

$$= (2p - q) - \frac{2(1-p)^2}{p}$$

$$= 3 - 2/p.$$

Example 6.5.2.

Given the r.v. X with p.d.f.

$$f(x) = \begin{cases} \frac{1}{2}x, & 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

Find the p,d.f. of Y, whose values are related to those of X by means of the equation $y = x^2$.

Solusion:

Since $y = x^2$, then $\frac{dy}{dx} = 2x$,

or $\frac{dx}{dy} = \frac{1}{2x} = \frac{1}{\mp 2\sqrt{y}}$, $y \neq 0$

Since $0 < x < 2$, then we neglect the negative sign. By applying (6.5.1) , we get

$$h(y) = f(x) . \left| \frac{dx}{dy} \right|$$

$$= f(\sqrt{y}) . \left| \frac{}{2\sqrt{y}} \right|$$

$$= \frac{\sqrt{y}}{2} . \frac{1}{2\sqrt{y}} = \frac{1}{4} \text{ , } 0 < y < 4$$

Therefore, the r.v.Y has a uniform distribution over (0.4).

To find the expected value of Y, we have

$$E(Y) = \int_0^4 y\, h(y)\, dy$$

$$= \int_0^4 \frac{y}{4} dy = \frac{y^2}{8} \Big]_0^4 = \frac{16}{8} = 2$$

The same result can be obtained if we apply (6.5.3) as

$$E(Y) = \int_{-\infty}^{\infty} g(x) f(x)\, dx$$

$$= \int_0^2 x^2 . \frac{1}{2} \times dx = \frac{x^4}{8} \Big]_0^2 = \frac{16}{8} = 2$$

Theorem 6.5.1.

Let X be a r.v. , then for any constants a and b, we have
$E(aX + b) = a E(X) + b$

Proof.

We prove this theorem for a continuous case. Let X be a continuous r.v. with p,d.f. f , then by Proposition (6.5.1.) we get

$$E = (aX + b) = \int_{-\infty}^{\infty} (ax + b) f(x)\,dx$$

$$= a \int_{-\infty}^{\infty} x f(x)\,dx + b \int_{-\infty}^{\infty} f(x)\,dx$$

$$= a E(X) + b \cdot 1$$

$$= a E(X) + b$$

Theorem 6.5.2.

Let X be a r.v. and for any constants a and b , we have

i. $E[ag(X) + b] = a E[g(x)] + b$.

ii. $E(a) = a$.

where g(X) is any function of X

Proof.

We prove this theorem for a discrete case.
Assume that X can take values $x_1, x_2, \ldots, x_n$ with respective probabilities $P(x_1)$, $P(x_2)$, ... , $P(x_n)$ then by Proposition (6.5.1), we have

i. $$E[ag(X) + b] = \sum_{i=1} [ag(x_i) + b] . P(x_i)$$

$$= a \sum_{i=1}^{n} g(x_i) P(x_i) + b \sum_{i=1}^{n} P(x_i)$$

$$= aE[g(X)] + b \cdot 1$$

$$= a E[g(X)] + b$$

ii. If g (X) = 1 and b=0, then E (a) = a.

6.6. Variance and Other Moments

Definition 6.6.1.

The *variance* of a r.v. X is denoted by Var (X) or σ_X^2 and is defined by

$$\text{Var}(X) = E[X - E(X)]^2 \qquad \text{............ (6.6.1)}$$

where E (X) is the expectation of X.
Equation (6.6.1.) can be written as

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

where

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx \text{ if X is a continuous}$$

$$= \sum_x x^2 P(x) \quad \text{if X is a discrete r. v.}$$

Definition 6.6.2.

The *r- th moment* about the origin is denoted by m_r, if it exists it is defined by

$$m_r = E(X^r), r = 1, 2, 3, \ldots$$

i. e.

$$m_r = \sum_x x^r P(x) \text{ if X is discrete}$$

$$= \int_{-\infty}^{\infty} x^r f(x)\,dx \text{ if X is continuous} \qquad \ldots (6{\cdot}6{\cdot}2)$$

The *r-th central moment* about the mean m = E (X) , is denoted by μ_r , if it exists it is defined by

$$\mu_r = E[X - m]^r, r = 1, 2, 3, \ldots$$

i. e.

$$\mu_r = \sum_x (x - m)^r P(x) \text{ if X is discrete}$$

$$= \int_{-\infty}^{\infty} (x - m)^r f(x)\, dx \text{ if X is continuous} \qquad \ldots (6\cdot6\cdot3)$$

Note that, if r=2, then μ_2 = Var (X).

The *standard deviation* of X is denoted by σ , and is defined as

$$\sigma = \sqrt{\text{Var}(X)}$$

Example 6.6.1.

Let X be a r.v. with a density function

$$f(x) = \frac{1}{b}, 0 < x < b$$
$$0 \quad \text{otherwise}$$

Then

$$m_1 = E(X) = \int_0^b \frac{x}{b}\, dx = \frac{b}{2},$$

$$m_2 = E(X^2) = \int_0^b \frac{x^2}{b}\, dx = \frac{b^2}{3}.$$

Then

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{b^2}{3} - \frac{b^2}{4} = \frac{b^2}{12}.$$

The 3 – rd moment about the origin is

$$m_3 = \int_0^b \frac{x^3}{b}\, dx = \frac{b^3}{4}.$$

Hence the 3 – rd central moment about the mean is

$$\mu_3 = E(x - m_1)^3 = \int_0^b \left(x - \frac{b}{2}\right)^3 . \frac{1}{b}\, dx$$

$$= \frac{\left(x - \frac{b}{2}\right)^4}{4b} \Bigg]_0^b = \frac{1}{4b}\left[\frac{b^4}{16} - \frac{b^4}{16}\right] = 0.$$

Example 6.6.2.

Let X be a r.v. with a p.m.f. given by

X = x	0	1	2	3
P (x)	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{3}{10}$	$\frac{4}{10}$

Then

$$m_1 = E(X) = \Sigma x\, P(x)$$

$$= 0\left(\frac{1}{10}\right) + 1\left(\frac{2}{10}\right) + 2\left(\frac{3}{10}\right) + 3\left(\frac{4}{10}\right)$$

$$= \frac{20}{10} = 2,$$

$$m_2 = E(X^2) = \Sigma x^2 P(x)$$

$$= 0\left(\frac{1}{10}\right) + 1\left(\frac{2}{10}\right) + 4\left(\frac{3}{10}\right) + 9\left(\frac{3}{10}\right)$$

$$= \frac{50}{10} = 2$$

Hence

$$\text{Var}(X) = E(X^2) - [E(X)]^2$$
$$= 5 - 4 = 1$$

The 4 – th moment about the origin is

$$m_4 = \Sigma x^4 \, P(x)$$

$$= 0\left(\frac{1}{10}\right) + 1\left(\frac{2}{10}\right) + 16\left(\frac{3}{10}\right) + 81\left(\frac{4}{10}\right)$$

$$= \frac{374}{10} = 37{\cdot}4{\cdot}$$

The 4- th central moment about the mean $m_1 = 2$ is

$$\mu_4 = E(X-2)^4$$
$$= \Sigma(x-2)^4 \, P(x)$$

$$= (0-2)^4\left(\frac{1}{10}\right) + (1-2)^4\left(\frac{2}{10}\right) + (2-2)^4\left(\frac{3}{10}\right)$$

$$+ (3-2)^4\left(\frac{4}{10}\right) = \frac{22}{10} = 2{\cdot}2{\cdot}$$