Lecture 7: Numerical Differentiation (Continuation of the Topic)

Taylor's Expansion:

Let $f(x)$ be a continuous and continuously differentiable function on an interval $[x, x+\Delta x]$ (i.e., the second derivative exists, the third derivative exists, and so on). Then, the value of the function f at a point $x+\Delta x$ can be approximated using what is called Taylor's Expansion as follows:

$$f(x+\Delta x) = \sum_{k=0}^{\infty} \frac{(\Delta x)^k}{k!} \frac{d^k f(x)}{\mathrm{dx}^k}$$

$$= \frac{(\Delta x)^0}{0!} f(x) + \frac{(\Delta x)^1}{1!} f'(x) + \frac{(\Delta x)^2}{2!} f''(x) + \frac{(\Delta x)^3}{3!} f'''(x) + ..$$

$$= f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2} f''(x) + \frac{(\Delta x)^3}{6} f'''(x) + ...$$

where $\frac{d^k f(x)}{d\mathrm{x}^k}$ represents the k-th derivative of the function at x. For example, when $k=3$, we have the third derivative

$$\frac{d^3 f(x)}{d\mathrm{x}^3} = f'''(x)$$

and so on.

$$k! = k \times (k-1) \times (k-2) \times (k-3) \times .... \times 2 \times 1$$

Comparison of Accuracy in Methods of Dividing Differences When Estimating the Derivative:

The following sections will discuss the reasons for the differences in estimating the derivative of a function between the three dividing difference methods discussed in the previous lecture (Forward Divided Difference, Backward Divided Difference, and Central Divided Difference).

Accuracy of Estimating the Derivative Using the Forward Divided Difference Method:

Assume $f(x)$ is a continuous and continuously differentiable function on an interval $[x, x+\Delta x]$, and Δx is relatively small. Then, using Taylor's Expansion, we can write the function f at the point $x+\Delta x$ as:

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) + \frac{(\Delta x)^3}{3!} f'''(x) + \ldots\ldots$$

$$\Rightarrow f(x+\Delta x) = f(x) + \Delta x f'(x) + O\left((\Delta x)^2\right)$$

$$\Rightarrow f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} - \frac{O\left((\Delta x)^2\right)}{\Delta x}$$

$$\Rightarrow f'(x) = \frac{f(x+\Delta x) - f(x)}{\Delta x} - O(\Delta x) \ldots\ldots\ldots(1)$$

Where the term involving $O(\Delta x)$ represents the **order of the error**. The Big O notation, $O(\Delta x)$), indicates that all the higher-order terms involving Δx tend to zero (or become so small that they can be neglected) as Δx approaches zero.

Note that equation (1) above is the same form used for estimating the derivative using the **Forward Divided Difference** method. The term

$O(\Delta x)$ here represents the error that we encounter when estimating the derivative using this method.

In this case, $O(\Delta x)$ the error decreases linearly with Δx, meaning that the error in the estimate of the derivative becomes smaller as Δx decreases. However, the rate of this decrease is proportional to Δx. This error term helps us quantify how accurate the method is and how the estimation improves when we use smaller values for $\Delta x \to 0$.

This should provide you with a clear understanding of how Taylor's Expansion relates to the forward divided difference method and its associated error.

Accuracy of Estimating the Derivative Using the Backward Divided Difference Method:

Assume $f(x)$ is a continuous and continuously differentiable function on an interval $[x - \Delta x, x]$, and Δx is relatively small. Then, using Taylor's Expansion, we can write the derivative of the function at the point $f(x)$ as:

$$f(x-\Delta x) = f(x) + (-\Delta x) f'(x) + \frac{(-\Delta x)^2}{2!} f''(x) + \frac{(-\Delta x)^3}{3!} f'''(x) + ...$$

$$\Rightarrow f(x-\Delta x) = f(x) - \Delta x f'(x) - O\left((\Delta x)^2\right)$$

$$\Rightarrow f'(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x} + \frac{O\left((\Delta x)^2\right)}{\Delta x}$$

$$\Rightarrow f'(x) = \frac{f(x) - f(x-\Delta x)}{\Delta x} + O(\Delta x) ....(2)$$

Thus, the error term in this approximation when using the Backward Divided Difference method is represented by $O(\Delta x)$, just like the Forward Divided Difference method. The error here is also linear in Δx.

Accuracy of Estimating the Derivative Using the Central Divided Difference Method:

Assume $f(x)$ is a continuous and continuously differentiable function on an interval $[x-\Delta x, x+\Delta x]$, and Δx is relatively small. Then, using Taylor's Expansion, we can write the function $f(x)$ at the points $[x-\Delta x, x+\Delta x]$ as:

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) + \frac{(\Delta x)^3}{3!} f'''(x) + \ldots\ldots$$

$$f(x-\Delta x) = f(x) + (-\Delta x) f'(x) + \frac{(-\Delta x)^2}{2!} f''(x) + \frac{(-\Delta x)^3}{3!} f'''(x) + \ldots\ldots$$

$$\Rightarrow f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) - \frac{(\Delta x)^3}{3!} f'''(x) \pm \ldots\ldots$$

By subtracting the two equations above, we get:

$$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x f'(x) + 2 \times O\left((\Delta x)^3\right)$$

$$\Rightarrow f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{2 \times O\left((\Delta x)^3\right)}{2\Delta x}$$

$$f'(x) = \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - O\left((\Delta x)^2\right) \ldots\ldots(3)$$

Thus, the error term in this approximation when using the Central Divided Difference method is represented by $O\left((\Delta x)^2\right)$, which means that the error decreases quadratically with Δx.

Notice that this is the same equation used for estimating the derivative using the Central Divided Difference method. The error here represents the amount of error we encounter when estimating the derivative with this method.

From the above, it is clear that the error when estimating the derivative using the Forward Divided Difference and Backward Divided Difference methods is of order $O(\Delta x)$, meaning it decreases linearly with Δx. On the other hand, the error when estimating the derivative using the Central Divided Difference method is of order $O\left((\Delta x)^2\right)$, meaning it decreases much faster and more accurately compared to the other two methods.

Thus, the estimation accuracy using the Central Divided Difference method is much better than that of the other two methods. This can be observed by comparing the results of estimating the derivative using all three methods on the example provided in Lecture 5 as shown below:

Example:

Estimate the value of the derivative of the following function using the three dividing difference methods that we discussed in the sixth lecture when $x = 3$

Δx	(FDD) $f'(x)$	(BDD) $f'(x)$	(CDD) $f'(x)$
0.1	291.3571	250.7734	271.0652
0.05	280.4363	260.173	270.3046
0.025	275.1787	265.0506	270.1147
0.01	272.0869	268.0361	270.0615

:

Notice that the true value of the derivative is:

$$f'(x) = 2\frac{d}{dx}e^{1.5x}$$

$$= 2(1.5e^{1.5x}) = 3e^{1.5x}$$

$$f'(3) \quad = 270.0514$$

It is clear that the Central Divided Difference method yields results that are very close to the true value compared to the other two methods, regardless of the value of Δx.