Herein, we give a formula which is useful to approximate n!, where n is large. It is called *Stirling's formula* :

$$n! \simeq \sqrt{2\pi n}\left(\frac{n}{e}\right)^{n}$$

The symbol $\simeq$ means approximately equal to. Where e is the base of natural logarithm, $e \simeq 2.7183$. For example,

$$10! \simeq \sqrt{2\left(\frac{22}{7}\right)(10)}\left(\frac{10}{2{\cdot}7183}\right)^{10}$$

$$\simeq 7{\cdot}9282\,(3{\cdot}67876)^{10}$$

$$= 3599060{\cdot}5$$

Theorem 2.3.2 - (Binomial Theorem).

$$(a+b)^{n} = \sum_{r=0}^{n} C(n,r).\,a^{r} b^{n-r} \qquad \ldots (2{\cdot}3{\cdot}3)$$

Proof

We shall prove the theorem by induction. The theorem is true for n=1 and 2. Assume it is true for n=k, i.e.

$$(a+b)^{k} = \sum_{r=0}^{k} C(k,r)\,a^{r} b^{k-r}$$

and we shall show that it is true for n=k+1. Thus

$$(a+b)^{k+1} = (a+b)(a+b)^{k}$$

$$= (a+b)\left[\sum_{r=0}^{k} C(k,r)\,a^{r} b^{k-r}\right]$$

$$
\begin{aligned}
&= (a+b)[C(k,0)a^0b^k + C(k,1)a^1b^{k-1} + \dots + C(k,k)a^kb^0] \\
&= C(k,0)a^0b^{k+1} + [C(k,0) + C(k,1)]ab^k + [C(k,1) \\
&+ C(k,2)]a^2b^{k-1} + \dots + C(k,k)b^0a^{k+1} \\
&= b^{k+1} + C(k+1,1)ab^k + C(k+1,2)a^2b^{k-1} + \dots + a^{k+1}
\end{aligned}
$$

(by Theorem 2.3.1)

So the theorem holds for all n.

Lemma 2.3.1

For any positive integer n, we have

$$(1+x)^n = \sum_{r=0}^{n} C(n,r)\,x^r$$

Proof:

In Theorem 2.3.2, putting a=x, b=1, hence the lemma follows.

By using Lemma 2.3.1, we can show that

$$\sum_{r=0}^{n} (-1)^r C(n,r) = 0 \text{ ,}$$

i. e. $C(n,0) - C(n,1) + C(n,2) - \dots + (-1)^n C(n,n) = 0.$

Theorem 2.3.3

$$\sum_{r=0}^{k} C(m,r) \, . \, C(n,k-r) = C(m+n,k)$$

We can prove Theorem 2.3.3 by using the same technique as in the proof of binomial theorem.

Note that in the application of Theorem 2.3.3, it is not necessary that m and n are greater than k.

For example, for m=3, n=4 and k=5, we have $C(3,0).\ C(4,5) + C(3,1).\ C(4,4) + C(3,2).\ C(4,3) + C(3,3)\ C(4,2) + C(3,4)\ C(4,1) + C(3,5).\ C(4,0) = C(7,5)$

Since $C(n,r) = 0$ when $r \geq n$, then the first, fourth and the fifth terms are zero, so the equation reduces to

$$C(3,1)\ C(4,4) + C(3,2)\ C(4,3) + C(3,3)\ C(4,2) = 3+3.4+1.6 = 21 = C(7,5).$$

2.4 Multinomial Expansion

Assume that there are n=7 students and that we wish to form 3 groups; 2 in the first, 3 in the second and 2 in the third group. Let $n_1 = 2$, $n_2 = 3$ and $n_3 = 2$ indicate the numbers in the groups. Then

$$n_1 + n_2 + n_3 = n = 7$$

There are $C(n, n_1) = C(7,2) = 21$ different ways of selecting the first group of 2 students. After that there are $C(n-n_1, n_2) = C(5,3) = 10$ different ways of selecting the second group of 3 student. Finally, there are $C(n - n_1 - n_2, n_3) = C(2,2) = 1$ way of selecting the remaining group.

By using the fundamental principle of counting, the total number of ways of selecting these three groups will be

$$C(n, n_1) . C(n - n_1, n_2) . C(n - n_1 - n_2, n_3)$$

$$= \frac{n!}{n_1!(n - n_1)!} \cdot \frac{(n - n_1)!}{n_2!(n - n_1 - n_2)!} \cdot \frac{(n - n_1 - n_2)!}{n_3!\,0!}$$

$$= \frac{n!}{n_1!\,n_2!\,n_3!} = \frac{7!}{2!\,3!\,2!} = 210.$$

In general, if we have n elements, and let $n_1, n_2, \ldots, n_k$ be positive integers with $n_1 + n_2 + \ldots + n_k = n$, then there exist.

$$\frac{n!}{n_1!\, n_2! \ldots n_k!}$$

different ordered partitions of n objects into these k groups, and are denoted by $\binom{n}{n, n_2, \ldots, n_k}$ or $C(n;\, n_1, n_2, \ldots, n_k)$ and is written as

$$C(n; n_1, n_2, \ldots, n_k) = \frac{n!}{n_1!\, n_2! \ldots n_k!} \qquad \ldots (2{\cdot}4{\cdot}1)$$

Equation (2.4.1) is reffered to as a *multinomial coefficient* in view of the following theorem which generalizes the binomial theorem

Theorem 2.4.1

$$(a_1 + a_2 + \ldots + a_k)^n = \Sigma\, C(n; n_1, n_2, \ldots, n_k)\, a_1^{n_1} a_2^{n_2} \ldots a_k^{n_k} \quad \ldots (2{\cdot}4{\cdot}$$

$$\ldots (2{\cdot}4{\cdot}2)$$

The summation is over all possible sets of integers $(n_1, n_2, \ldots, n_k)$ such that $0 \leq n_i \leq n,\ i = 1, 2, \ldots, n$ and $n_1 + n_2 + \ldots + n_k = n$. Equation (2.4.2) is called the multinomial theorem.

There are C(n+k-1, n) terms in the expansion of equation (2.4.2).

Example 2.4.1

1. Find the number of ways in which 10 rolls of a die can yield 2 ones, 3 twos, no threes, 1 four, 2 fives, and 2 sixes,
   We have·
   $n = 10, n_1 = 2, n_2 = 3, n_3 = 0, n_4 = 1, n_5 = 2, n_6 = 2,$
   $$C(10; 2, 3, 1, 2, 2) = \frac{10!}{2!\,3!\,1!\,2!\,2!} = 529200 \text{ ways.}$$

2. Find the number of ways in which one A, three B's, two C's, and one F can be distributed among 7 students taking a course in statistics.
   We have
   $n = 7, n_1 = 1, n_2 = 3, n_2 = 2, n_4 = 1$, then
   $$C(7; 1, 3, 2, 1) = \frac{7!}{1!\,3!\,2!\,1!} = 420 \text{ ways}$$

Example 2.4.2

Using multinomial theorem to evaluate $(x_1 + x_2 + x_3)^2$ ..

We have $(x_1 + x_2 + x_3)^2 = \Sigma\, C(2; n_1, n_2, n_3)\, x_1^{n_1} x_2^{n_2} x_3^{n_3}$

$$= C(2; 2, 0, 0)\, x_1^2 x_2^0 x_3^0 + C(2; 0, 2, 0)\, x_1^0 x_2^2 x_3^0 + C(2; 0, 0, 2)\, x_1^0 x_2^0 x_3^2$$
$$+ C(2; 1, 1, 0)\, x_1^1 x_2^1 x_3^0 + C(2; 1, 0, 1)\, x_1^1 x_2^0 x_3^1 + C(2; 0, 1, 1)\, x_1^0 x_2^1 x_3^1$$
$$= x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 .$$