Example 6.6.3.

Let X be a r.v. having a binomial distribution with paramenters n and p, then we have E (X) = np,

To find $E(X^2)$, write $x^2 = x(x-1) + x$, then

$$E(X^2) = E[X(X-1)+X]$$
$$= E[X(X-1)] + E[X].$$

Now

$$E[X(X-1)] = \sum_{x=0}^{n} x(x-1).C(n,x)\,p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{x(x-1)\,n!}{x!(n-x)!}\,p^x q^{n-x}$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!}\,p^{x-2} q^{n-x}$$

$$\left(\text{let } n-2=m,\quad x-2=y\right.$$

$$= n(n-1)p^2(p+q)^m$$
$$= n(n-1)p^2.$$

Therefore

$$E(X^2) = n(n-1)p^2 + np,$$

hence

$$\mathrm{Var}(X) = E(X^2) - [E(X)]^2$$
$$= n(n-1)p^2 + np - n^2p^2$$
$$= npq.$$

Example 6. 6. 4.

Let X be a r. v. have a geometric distribution with parameter p, then we have

$$p(x) = pq^{x-1}, x = 1, 2, \ldots$$

and

$$E(X) = 1/p$$

To find the variance of X , write $x^2 = x(x-1) + x$, thus

$$E(X^2) = \sum_{x=2}^{\infty} x(x-1)pq^{x-1} + \sum_{x=1}^{\infty} xpq^{x-1}$$

$$= \Sigma x(x-1)pq^{x-1} + E(X).$$

We have

$$\sum_{x=1}^{\infty} xq^{x-1} = \frac{1}{(1-q)^2}.$$

Then

$$\frac{d}{dq}(\Sigma xq^{x-1}) = \frac{d}{dq}\left(\frac{1}{(1-q)^2}\right)$$

$$\sum_{x=2}^{\infty} x(x-1)q^{x-2} = \frac{2}{(1-q)^3}.$$

Therefore

$$E(X^2) = pq\sum_{x=2}^{\infty} x(x-1)q^{x-2} + 1/p$$

$$= \frac{2pq}{(1-q)^3} + \frac{1}{p} = \frac{2q}{p^2} + \frac{1}{p},$$

hence

$$\operatorname{Var}(X) = E(X^2) - [E(X)]^2$$

$$= \frac{2q}{p^2} \qquad \frac{1}{p} - \frac{1}{p^2}$$

$$= \frac{2q + p - 1}{p^2} = \frac{q}{p^2} .$$

Example 6. 6. 5.

Let X be a r. v. have a normal distribution with parameters μ and σ^2.

$\operatorname{Var}(X) = E(X - E(X))^2 = E(X - \mu)^2$ be cause $E(X)$
$= \mu$.

$$\operatorname{Var}(X) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x - \mu)^2 e^{-\frac{1}{2\sigma^2}(X-\mu)^2} dx$$

$$\text{let } \frac{x - \mu}{\sigma} = y \Longrightarrow dx = \sigma dy$$

$$\operatorname{Var}(X) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \sigma^2 y^2 e^{-\frac{1}{2}y^2} . \sigma dy$$

$$= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 e^{-\frac{1}{2}y^2} dy$$

$$= \frac{2\sigma^2}{\sqrt{2\pi}} \int_{0}^{\infty} y^2 e^{-\frac{1}{2}y^2} dy.$$

Now , let $\frac{y^2}{2} = u$

$$\frac{2y}{2} dy = du \Longrightarrow dy = \frac{du}{\sqrt{2u}} ,$$

therefore

$$\operatorname{Var}(X) = \frac{2\sigma^2}{\sqrt{2\pi}} \int_0^\infty 2u \,.\, e^{-u} \frac{du}{\sqrt{2u}}$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty u^{\frac{1}{2}} e^{-u}\, du$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right) \qquad \text{by equation } (5{\cdot}6{\cdot}4)$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \sigma^2 \left(\text{since } \Gamma \frac{1}{2} = \sqrt{\pi}\right)$$

Theorem 6.6.1.

Let X be a r.v. and for any constants a and b , we have
i. $\operatorname{Var}[aX + b] = a^2 \operatorname{Var}(X)$.
ii. $\operatorname{Var}[a] = 0$.

Proof: We have
i. $\operatorname{Var}[aX + b] = E[aX + b - E(aX + b)]^2$

$$= E[aX + b - (a E(X) + b)]^2 \text{ by Theorem (6.5.1)}$$
$$= E[aX - a E(X)]^2$$
$$= E[a(X - E(X))]^2$$
$$= a^2 E(X - E(X))^2 = a^2 \operatorname{Var}(X).$$

ii. $\operatorname{Var}(a) = E[a - E(a)]^2$
$= E[a - a]^2 = o$ by Theorem (6.5.2)

6.7. Some Moment Inequalities

1. Markov's Inequality

If X is a random variable that takes only nonnegative values, then for any constant $a > 0$,

$$P(X \geq a) \leq \frac{E(X)}{a} \qquad \dots (6{\cdot}7{\cdot}1)$$

Proof.

We shall prove this inequality for continuous case. Let the density of X be f, then we have

$$E(X) = \int_0^\infty x f(x)\,dx = \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty x f(x)\,dx \qquad \text{because} \int_0^{a} x f(x)\,dx \geq 0$$

$$\geq a \int_a^\infty f(x)\,dx$$

$$= a\,P(X > a).$$

Hence

$$P(X \geq a) \leq \frac{E(X)}{a}$$

A generalization form of Markov's inequality is : if g(x) is a nonnegative function of a r.v. X. Then for every $a > 0$, we have

$$P(g(X) \geq a) \leq \frac{E(g(X))}{a} \qquad \dots (6{\cdot}7{\cdot}2)$$

For the proof see Gupta and Kapoor (1982)

2. Chebyshev's Inequality

If X is a r.v. with a finite mean m and variance σ^2 then for any $\varepsilon > 0$, we have

$$P\{|X - m| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2} \qquad \text{... (6·7·3)}$$

or

$$P\{|X - m| < \varepsilon\} \geq 1 - \frac{\sigma^2}{\varepsilon^2} \qquad \text{... (6·7·4)}$$

Proof :

Let X be a continuous r.v. with density function f, then

$$\sigma^2 = \mathrm{Var}(X) = E(X - E(X))^2$$

$$= E(X - m)^2$$

$$= \int_{-\infty}^{\infty} (x - m)^2 f(x)\, dx$$

$$= \int_{-\infty}^{m-\varepsilon} (x - m)^2 f(x)\, dx + \int_{m-\varepsilon}^{m+\varepsilon} (x - m)^2 f(x)\, dx$$

$$+ \int_{m+\varepsilon}^{\infty} (x - m)^2 f(x)\, dx$$

$$\geq \int_{-\infty}^{m-\varepsilon} (x - m)^2 f(x)\, dx + \int_{m+\varepsilon}^{\infty} (x - m^2) f(x)\, dx$$

For the first integral $x \leq m - \varepsilon \Longrightarrow x\text{-}m \leq -\varepsilon$ and for the second integral $x \geq m + \varepsilon \Longrightarrow x\text{-}m \geq \varepsilon$ then

$$\sigma^2 \geq \varepsilon^2 \int_{-\infty}^{m-\varepsilon} f(x)\,dx + \varepsilon^2 \int_{m+\varepsilon}^{\infty} f(x)\,dx$$

$$= \varepsilon^2 P\{X \leq m - \varepsilon\} + \varepsilon^2 P\{X \geq m + \varepsilon\}$$

$$= \varepsilon^2 P\{X - m \leq -\varepsilon\} + \varepsilon^2 P\{X - m \geq \varepsilon\}$$

$$= \varepsilon^2 P\{|X - m| \geq \varepsilon\}$$

Therefore

$$P|X - m| \geq \varepsilon\} \leq \frac{\sigma^2}{\varepsilon^2}$$

For the discrete case the proof can be adapted by changing the integration sign by summation.

Remarks :

1. The Chebyshev's inequality can be obtained if we let g (X) = $(X - m)^2$ and a = ε^2 in equation (6.7.2) we get

$$P\{(X - m)^2 \geq \varepsilon^2\} \leq \frac{E(X - m)^2}{\varepsilon^2} = \frac{Var(X)}{\varepsilon^2}$$

hence

$$P\{|X - m| \geq \varepsilon\} < \frac{\sigma^2}{\varepsilon^2}$$

2. Since P $\{|X - m| \geq \varepsilon\}$ +P$\{|x-m| < \varepsilon\} = 1$, then by applying equation (6.7.3) we get equation (6.7.4).

3. The Chebyshev's inequality can also be written as

$$P\{|X - m| \geq a\sigma\} \leq \frac{1}{a^2}.$$

Fig (6.1) gives a graphical representation of the Chebyshev's inequality

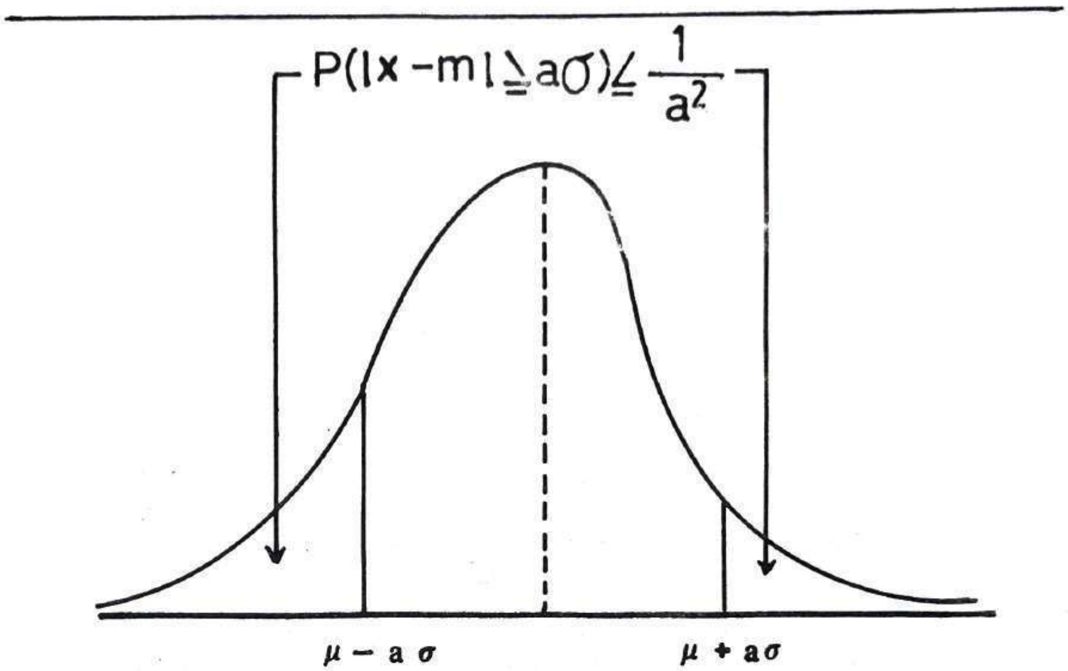

Fig. 6.1 Chebyshev's inequality

3. Jensen's Inequality

Let X be a r.v. with finite mean E (X), and let f be a convex function, then

$$E[f(X)] \geq f[E(X)] \qquad \ldots (6{\cdot}7{\cdot}5)$$

Before we give the proof of Jensen's inequality, we define the convex function as :

Let f be a real - valued function defined on an interval I ($I \subset R$). f is said to be *convex* if for every pair of points x_1 and x_2 of I,

$$f\left(\frac{x_1 + x_2}{2}\right) \leq \frac{1}{2} f(x_1) + \frac{1}{2} f(x_2)$$

Proof :

Consider a tangent line to the function f at the point $(x_0, f(x_0))$ (see Fig. 6.2), let the equation of the tangent be

$y = a x + b.$
Since f is convex, then
$f(x) \geq a x + b$
for all x ; hence
$f(x_0) \geq a x_0 + b$ -

Thus
$$E[f(x_0)] \geq E[ax_0 + b]$$
$$= a E(x_0) + b$$
$$= f(E(x_0))$$
when $E(X) = x_0$, thus (6.7.5) follows.

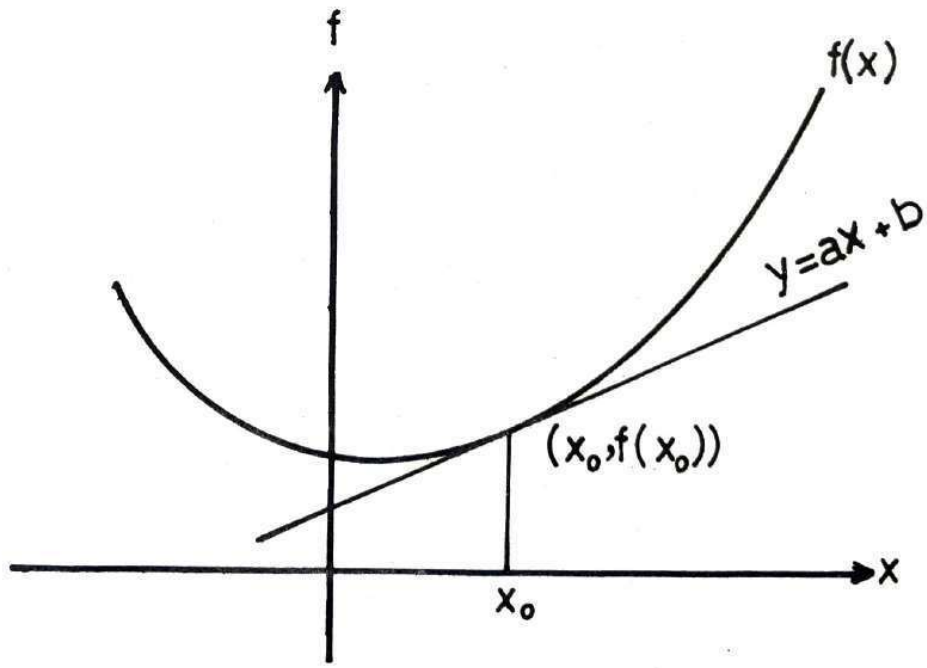

Fig. 6.2 Jensen's inequality.