Lecture 8 – Numerical Analysis 2

High-Order Numerical Derivation:

In this section, we will discuss how to estimate the second-order derivative numerically. Let's assume that a function $f(x)$ is continuous and continuously differentiable on an interval $[x-\Delta x, x+\Delta x]$. In this case, we can use what's called the **Taylor Expansion** to derive an estimated formula for the second derivative of the function $f(x)$. The Taylor series expansion is given as:

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) + \frac{(\Delta x)^3}{3!} f'''(x) + \frac{(\Delta x)^4}{4!} f''''(x) + ...$$

$$f(x-\Delta x) = f(x) + (-\Delta x) f'(x) + \frac{(-\Delta x)^2}{2!} f''(x) + \frac{(-\Delta x)^3}{3!} f'''(x) + \frac{(-\Delta x)^4}{4!} f''''(x) + ...$$

$$\Rightarrow f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) - \frac{(\Delta x)^3}{3!} f'''(x) + \frac{(\Delta x)^4}{4!} f''''(x) + ...$$

Now, by adding the two equations above $f(x+\Delta x) + f(x-\Delta x)$, we obtain:

$$f(x+\Delta x)+f(x-\Delta x)=2f(x)+2\frac{(\Delta x)^2}{2!}f''(x)+2\frac{(\Delta x)^4}{4!}f''''(x)+\ldots$$

$$\Rightarrow f(x+\Delta x)+f(x-\Delta x)=2f(x)+(\Delta x)^2 f''(x)+O\left((\Delta x)^4\right)$$

$$\Rightarrow f''(x)=\frac{f(x+\Delta x)+f(x-\Delta x)-2f(x)}{(\Delta x)^2}-\frac{O\left((\Delta x)^4\right)}{(\Delta x)^2}$$

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$$\Rightarrow f''(x)=\frac{f(x+\Delta x)+f(x-\Delta x)-2f(x)}{(\Delta x)^2}\ldots\ldots(4)$$

The last formula (Equation 4) is the one we will use to estimate the second derivative of the function $f(x)$. The error that we will encounter when estimating the second derivative is equal to $O\left((\Delta x)^2\right)$.

Example:

Estimate the value of the second derivative of the function $f(x)=2e^{1.5x}$, and then calculate the absolute and relative error of the estimate.

Solution: Applying Equation 4 above with $\Delta x = 0.1$ and x=3:

$$f'(x=3) \approx \frac{f(x+\Delta x)+f(x-\Delta x)-2f(x)}{(\Delta x)^2}$$

$$= \frac{f(3+0.1)+f(3-0.1)-2f(3)}{(0.1)^2}$$

$$= \frac{f(3.1)+f(2.9)-2f(3)}{0.01}$$

$$= \frac{2e^{1.5(3.1)}+2e^{1.5(2.9)}-2\left(2e^{1.5(3)}\right)}{0.01} = 405.8373$$

The estimated value of the second derivative is (405.8372). when $\Delta x = 0.1$ and x=3

The true value of the second derivative is:

$$f'(x) = 2\frac{d}{dx}e^{1.5x}$$

$$= 2\left(1.5e^{1.5x}\right) = 3e^{1.5x}$$

$$f'(x=3) = 4.5e^{1.5\times 3} = 405.0771$$

As can be observed, the estimated value is very close to the true value of the second derivative.

In this case, the absolute error is calculated as follows:

$$AbsoluteError = |405.0771 - 405.8372| = 0.7601$$

Homework:

Estimate the value of the second derivative of the following function when x=2, $\Delta x = 0.1$:

$$f(x) = \ln(2x-1)$$

Then, find the absolute error of the estimated solution compared to the real solution, noting that the second real derivative is:

$$f''(x) = \frac{-4}{(2x-1)^2}$$