Proof :

Consider a tangent line to the function f at the point $(x_0, f(x_0))$ (see Fig. 6.2), let the equation of the tangent be

$y = a x + b$.

Since f is convex, then

$f(x) \geq a x + b$

for all x ; hence

$f(x_0) \geq a x_0 + b$ -

Thus

$$E[f(x_0)] \geq E[a x_0 + b]$$
$$= a E(x_0) + b$$
$$= f(E(x_0))$$

when $E(X) = x_0$, thus (6.7.5) follows.

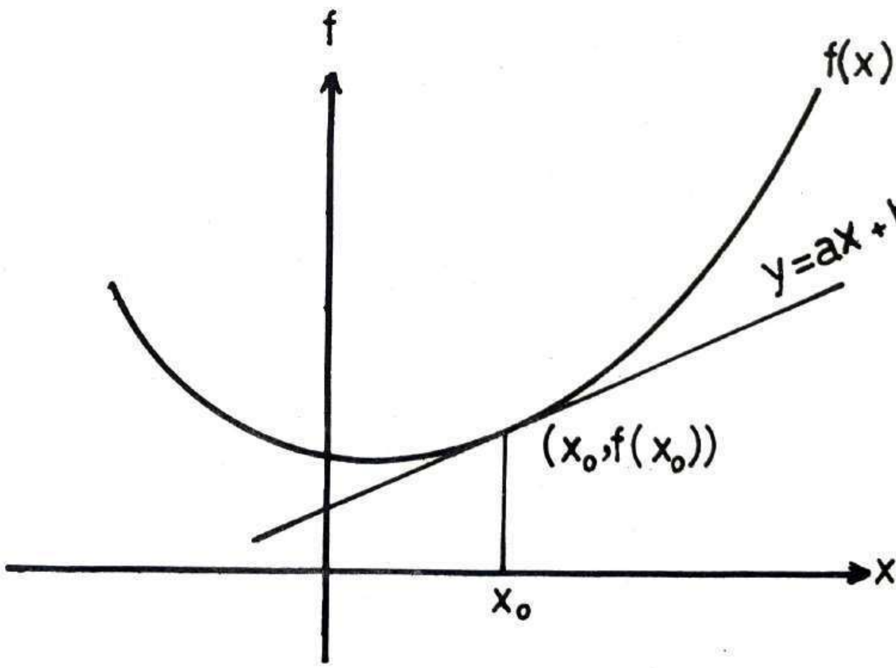

Fig. 6.2 Jensen's inequality.

Example 6.7.1.

An unbiased die is thrown 480 times, find the lower bound for the probability of getting 60 to 100 sixes.

Solution.

Let X be the number of sixes obtained , then X has a binomial distribution with parameters n = 480 and P = 1/6. Thus E (X) = np = 480 (1/6) = 80, and Var (X) = σ^2 = n p q = 480 (1/6) (5/6)
= 66.666.
Use Chebyshev's inequality (6.7.4), i.e.

$$P\{|X - E(X)| < \varepsilon\} \geq 1 - \frac{Var(X)}{\varepsilon^2},$$

We get

$$P\{-\varepsilon < X - 80 < \varepsilon\} \geq 1 - \frac{66{\cdot}666}{\varepsilon^2}$$

or

$$P\{80 - \varepsilon < X < 80 + \varepsilon\} \geq 1 - \frac{66{\cdot}666}{\varepsilon^2}.$$

Now. Compare the last inequality with P {60 < X < 100} , we get $\varepsilon = 20$.

Therefore,

$$P\{60 < X < 100\} \geq 1 - \frac{66{\cdot}666}{400}$$

$$= 1 - 0{\cdot}166 = \underline{0{\cdot}834}$$

Example 6.7.2.

Two unbiased dice are thrown If X is the sum of the numbers showing up, prove that

$$P\{|X-7|\geq 3\} \leq \frac{35}{54}.$$

Solution

By Chebyshev's inequality, for $\varepsilon > o$, we have

$$P\{|X-m|\geq \varepsilon\} \leq \frac{Var(X)}{\varepsilon^2}$$

We have

$$m = E(X) = \Sigma x P(x)$$

$$= 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + 4\left(\frac{3}{36}\right) + 5\left(\frac{4}{36}\right)$$

$$+ 6\left(\frac{5}{36}\right) + 7\left(\frac{6}{36}\right) + 8\left(\frac{5}{36}\right) + 9\left(\frac{4}{36}\right)$$

$$+ 10\left(\frac{3}{36}\right) + 11\left(\frac{2}{36}\right) + 12\left(\frac{1}{36}\right) = 7,$$

and

$$E(X^2) = \Sigma x^2 P(x)$$

$$= \frac{1}{36}[1(2)^2 + 2(3)^2 + 3(4)^2 + 4(5)^2 + 5(6)^2 + 6(7)^2$$

$$+ 5(8)^2 + 4(9)^2 + 3(10)^2 + 2(11)^2 + 1(12)^2]$$
$$= 329/6.$$

Thus

$$\text{Var}(X) = E(X^2) - (E(X))^2$$

$$= \frac{329}{6} - (7)^2 = 35/6.$$

We have m = 7, σ^2 = 35/6, and ε = 3, therefore,

$$P\{|X-7| > 3\} \leq \frac{35/6}{9} = \frac{35}{54}.$$

But the actual probability is

$$\begin{aligned} P\{|X-7| \geq 3\} &= P\{X-7 \geq 3 \text{ or } X-7 \leq -3\} \\ &= P\{X \geq 10 \text{ or } X \leq 4\} \\ &= P\{X \geq 10\} + P\{X \leq 4\} \\ &= \frac{3}{36} + \frac{2}{36} + \frac{1}{36} + \frac{1}{36} + \frac{2}{36} + \frac{3}{36} \\ &= \frac{12}{36} = \frac{1}{3}. \end{aligned}$$

Example 6.7.3.

Suppose it is known that the number of items produced in a factory during a week is a random variable with a mean of 600.

1. What can be said about the probability that this week's production will exceed 720?
2. If the variance of a week's production is known to equal 20, then what can be said about the probability that this week's production will be between 590 and 610?

Solution

Let X denote the number of items produced in a given week.

1. By Markov's inequality, we have

$$P\{X \geq a\} \leq \frac{E(X)}{a},$$

i. e.

$$P\{X > 720\} \leq \frac{600}{720} = \frac{5}{6}.$$

2. $P\{590 < X < 610\} = P\{-10 < X - 600 < 10\}$

$$\geq 1 - \frac{Var(X)}{\varepsilon^2}.$$

$$\geq 1 - \frac{20}{(10)^2} = 1 - \frac{20}{100} = \frac{4}{5}.$$

6.8 Problems

1. Let X be a r.v. with p.m.f. P(x) given by

X = x	-4	-3	-2	-1	0	1	2
P(x)	k	2k	3k	2k	k	4k	3k

a. Find k
b. Find E(X) and Var (X).
c. Let $Y = X^2$, find E(Y) and Var (Y)

2. In a lottery there are 250 prizes of 5 I.D., 150 prizes of 10 I.D., 70 prizes of 25 I.D. . 50 prizes of 50 I.D., 30 prizes of 100 I.D. and 10 prizes of 200 I.D. Assuming that 10000 tickets are to be issued and sold, what is a fair price to pay for the ticket?

3. Find EX and Var (X) of a discrete r.v. whose p.m.f. is given by

a. $P(x) = 2(1/3)^x$; $x = 1, 2, 3, \ldots,$

b. $P(x) = x/15$ for $x = 1, 2, 3, 4, 5$

c. $P(x) = e^{-3}3^x/x!$, $x = 0, 1, 2, \ldots$

d. $P(x) = C(5, x)\left(\frac{1}{4}\right)^x\left(\frac{3}{4}\right)^{5-x}$, $x = 0, 1, \ldots, 5$

4. Find E(X) and Var (X) of a continuous r.v. X whose p.d.f. is given by

a. $f(x) = 3e^{-3x}$, $x \geq 0$
   0 otherwise.

b. $f(x) = 1$, $0 < x < 1$
   0 otherwise.

c. $f(x) = \dfrac{4x(9 - x^2)}{81}$, $0 \leq X \leq 3$
   0 otherwise.

d. $f(x) = 3x^2$, $0 < x < 1$
   0 otherwise.

5. Let X be a r.v. having a Poisson distribution with parameter m. Show that Var(X) = m.

6. Let X be a r.v. having an exponential distribution $f(x) = \lambda e^{-\lambda x}$, $x > 0$. Show that Var (X) = $1/\lambda^2$.

7. Let X be a r.v. having a geometric distribution given by

$$P(x) = pq^x \quad , \quad x = 0, 1, 2, \ldots$$

Find the mean and variance of X.

8. Let Y be a r.v. with a uniform density

$$f(y) = \begin{cases} \dfrac{1}{\pi}, & -\dfrac{\pi}{2} < y < \dfrac{\pi}{2} \\ 0 & \text{otherwise} \end{cases}$$

Find Var (Y) , E (Y^3), E (Y − 2 $)^2$ and Var (5Y+2).

9. Find μ_3 and μ_4 for the r.v. Z whose p.m.f. is given by

$$P(z) = \frac{1}{2} \quad \text{for } z = -2 \text{ and } z = 2.$$

10. Express Var (Y) in terms of Var (X) if the values of Y are related to those of X by means of the equation

a. y= x+c ;
b. y= cx +b ,
where c and b are constants.

11. Let Z be a r.v. with a p.d.f. given by

$$f(z) = \begin{cases} \dfrac{1}{z}, & 1 < z < e \\ 0 & \text{otherwise} \end{cases}$$

Find Var (Z).

12. Let X be a r.v. with mean μ and variance σ^2 . Show that the r.v. $Z = \dfrac{X - \mu}{\sigma}$ has mean zero and variance 1.

13. Check whether the mean and the variance of the following distributions exist

a. $f(x) = \begin{cases} 2x^{-3}, & x > 1 \\ 0 & \text{otherwise} \end{cases}$

b. $f(x) = \dfrac{\pi}{\pi^2 + x^2}, \quad -\infty < x < \infty .$

$\dfrac{1}{1-q}$

$(1-q)^{-1}$

$-1(1+q)(-1)$

14. A r.v. X has a density function given by

$$f(x) = \begin{cases} 4e^{-4x} & , x \geq 0. \\ 0 & \text{otherwise} \end{cases}$$

Let $Y=X^2$, find h(y) and hence calculate E(Y).

15. A r.v. X has a normal distribution,

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Let $Y = e^X$, find h(y) and hence calculate E(Y).

16. Let X be a r.v. having a Poisson distribution with parameter m. Let Y=X+1, find the p.m.f. of Y and hence find E(Y).

17. A r.v. X has a p.d.f. given by

$$f(x) = \begin{cases} 2e^{-2x} & , x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Use Chebyshev's inequality to obtain the upper bound on $P\{ |X-E(X)| > 1\}$ and compare it with the actual value.

18. Prove Chebyshev's inequality when X is a discrete r.v.

19. A r.v. X has the p.d.f.

$$f(x) = \frac{1}{2} e^{-|x|}, \quad -\infty < x < \infty$$

a. Find $P\{|X-\mu| > 2\}$.

b. Use Chebyshev's inequality to obtain an upper bound on $P\{|X-\mu| > 2\}$.