Lecture 9: Numerical Integration

Mathematical Definition of Integration:

Let $f(x)$ be a real function that is continuous on the interval $[a,b]$. Then, the integral of the function is given by:

$$\int_a^b f(x)dx = F(b) - F(b)$$

where F is the function resulting from integrating $f(x)$. The integral represents the area under the curve of the function between the points a,b . In some cases, finding the function F explicitly can be complex or even impossible. Therefore, we resort to estimating the value of the integral using numerical methods. Note that we are only talking about estimating definite integrals, as the theoretical formula for indefinite integration cannot be computed numerically. In the following sections, we will discuss some methods used to estimate the integral numerically.

Trapezoidal Rule:

To estimate the integral $\int_a^b f(x)dx$ we divide the interval $[a,b]$ into n equal subintervals, as follows:

$$[a,b] \Rightarrow [x_0, x_1],[x_1, x_2],[x_2, x_3]......[x_{n-1}, x_n]$$

$$a = x_0 < x_1 < x_2 < x_3 .... < x_{n-1} < x_n = b$$

The length of each interval is given by:

$$h=\frac{b-a}{n}=x_i-x_{i-1,}\quad for\ \text{i=1,2,3....,n}$$

$$\int_a^b f(x)dx=\int_{x_0}^{x_1} f(x)dx+\int_{x_1}^{x_2} f(x)dx+\int_{x_2}^{x_3} f(x)dx+......+\int_{x_{n-1}}^{x_n} f(x)dx$$

$$\Rightarrow\int_a^b f(x)dx=\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)dx$$

Each of the above partial integrals represents a portion of the area under the curve of the function. The shape of the area within each subinterval is approximately a trapezoid (as shown in Figure 1). Therefore, we can compute the area of the trapezoid as an estimate for the area under the curve. The area of a trapezoid is given by:

$$Area\ \text{of Trapezoid=}\left(\frac{b_1+b_2}{2}\right)h$$

where b_1 b_2 ,are the lengths of the parallel sides of the trapezoid, and h is the height (as shown in Figure 2).

Figure 1: The area under the function's curve and its estimation using trapezoids.

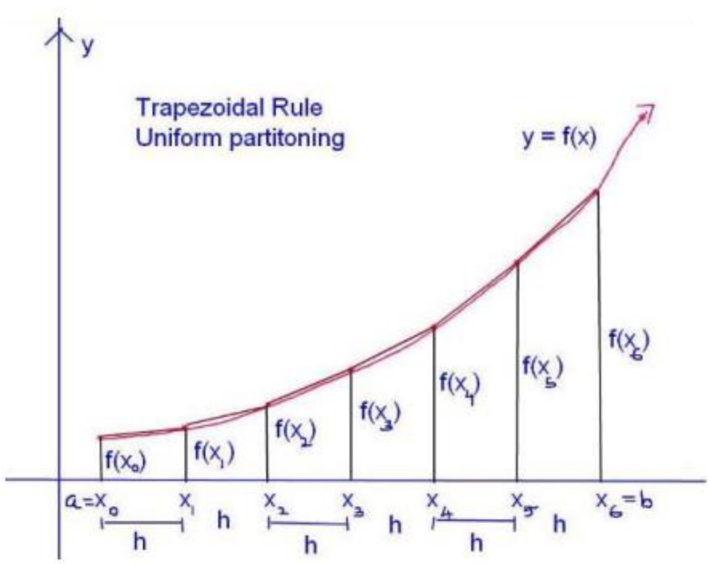
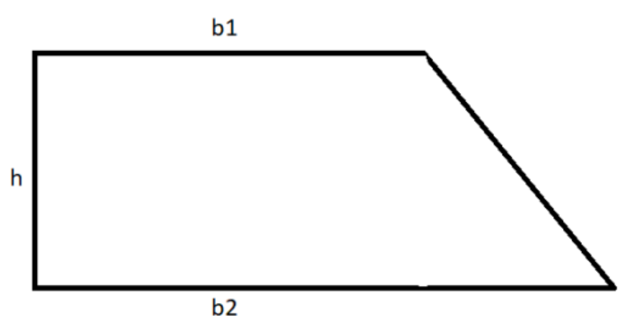


Figure 2: Right-angled trapezoid.



As shown in Figure 1, we can estimate each partial integral as follows:

$$\int_{x_i - 1}^{x_i} f(x)dx = \left(\frac{f(x_i - 1) + f(x_i)}{2} \right) h$$

Thus, as shown in Figure 1, we can estimate each partial integral as follows:

$$\int_a^b f(x)dx = \int_{x_0}^{x_1} f(x)dx + \int_{x_1}^{x_2} f(x)dx + \int_{x_2}^{x_3} f(x)dx + \dots\dots + \int_{x_{n-1}}^{x_n} f(x)dx$$

$$= \left(\frac{f(x_0) + f(x_1)}{2} \right) h + \left(\frac{f(x_1) + f(x_2)}{2} \right) h + \dots\dots + \left(\frac{f(x_{n-1}) + f(x_n)}{2} \right) h$$

$$\Rightarrow \int_a^b f(x)dx = \frac{h}{2}\Big(f(x_0) + 2\big(f(x_1) + f(x_2) + \dots\dots + f(x_{n-1}) \big) + f(x_n) \Big)$$

Example:

Estimate the integral $\int_{0.5}^{1} x^4 e^x dx$ using the trapezoidal method when $h = 0.1$.

Solution: We set the values $h = 0.1$, and therefore we have the $x_0, x_1, \dots . x_n$ following intervals:

$$[0.5,1] = [0.5,0.6],[0.6,0.7],[0.7,0.8],[0.8,0.9][0.9,1]$$

Substituting x_0, x_1, x_3, x_4, x_5 into the function $f(x) = x^4 e^x$, we get the following values for $f(x)$:

x	$a = x_0 = 0.5$	$x_1 = 0.6$	$x_2 = 0.7$	$x_3 = 0.8$	$x_4 = 0.9$	$b = x_5 = 1$
$f(x)$	0.1030	0.2361	0.4835	0.9116	1.6137	2.7183

The estimated integral value is calculated using the trapezoidal rule.

After calculating the sum, we find the estimated integral $\int_{0.5}^{1} x^4 e^x dx$ value to be:

$$\int_{0.5}^{1} x^4 e^x dx = \frac{h}{2}\left(f(x_0) + 2\left(f(x_1) + f(x_2) + f(x_3) + f(x_4) \right) + f(x_5) \right)$$

$$\int_{0.5}^{1} x^4 e^x dx = \frac{0.1}{2}\left(0.1030 + 2\left(0.2361 + 0.4835 + 0.9116 + 1.6137\right) + 2.7183\right)$$

$$= 0.4656$$

Note that the exact value of this integral $\int_{0.5}^{1} x^4 e^x dx$ can be found mathematically using the formula for the integral of the exponential function:

$$\int u dv = uv - \int v du$$

However, solving this indefinite integral analytically is very complex. The value above is computed numerically.

$$\int x^4 e^x dx = e^x \left(x^4 - 4x^3 + 12x^2 + 24x + 24\right) + c$$

Example:

Estimate the integral $\int_0^1 x^5 \sqrt{2 - x^3}\, dx$ according to the trapezoidal method when $n = 4$.

Solution:

Notice that:

$$h = \frac{b-a}{n} = \frac{1-0}{4} = 0.25$$

Thus, the values of x are as follows:

x	$a = x_0 = 0$	$x_1 = 0.25$	$x_2 = 0.5$	$x_3 = 0.75$	$b = x_4 = 1$
$f(x)$	0	0.0014	0.0428	0.2981	1

Now, we can estimate the integral using the trapezoidal method as follows:

$$\int_0^1 x^5 \sqrt{2 - x^3}\, dx = \frac{h}{2}\Big(f(x_0) + 2\big(f(x_1) + f(x_2) + f(x_3) + \big) + f(x_4) \Big)$$

$$= \frac{0.25}{2}\Big(0 + 2\big(0.0014 + 0.0428 + 0.2981\big) + 1\Big) = 0.2106$$

Thus, the estimated value of the integral is 0.2106 according to the trapezoidal method when $n = 4$.

Note that the above integral can be evaluated theoretically, but it requires very complex steps. The formula for the function after integration is as follows:

$$\int_0^1 x^5 \sqrt{2-x^3}\,dx = -\frac{2}{45}\left(2-x^3\right)^{\frac{3}{2}}\left(3x^3+4\right)+c$$

From the above examples, we can conclude that numerical integration is much easier than theoretical integration, especially when using computational programs to find the integral.