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Chapter Seven
Jointly Distributed Random Variables

Chapter Seven. Jointly Distributed Random Variables

7.1. Joint Distribution Function

In Chapters Five and Six we have considered the distribution function and the moments of one random variable. The defintions of the distribution function and the moments are easily generalized to two (or more) random variables. Here we consider the case when the two random variables are both either discrete or continuous.

Two random variables X and Y are said to be jointly distributed if they are defined on the same probability sample space S. The sample points consists of the orderd pairs (x,y). The range of each r.v. is a set of real numbers. The pair (x,y) represents a point in a plane, where

$X : S \to R_1$, where $X(w) = x$;
$Y : S \to R_2$, where $Y(w) = y$;
$(X,Y) : S \to R_1 \times R_2$ such that ; $(X,Y)(w) = (x,y)$

Thus (X,Y) maps S into the plane $R_1 \times R_2$ which is the cartesian product of two axes. This idea is represented in Fig 7.1

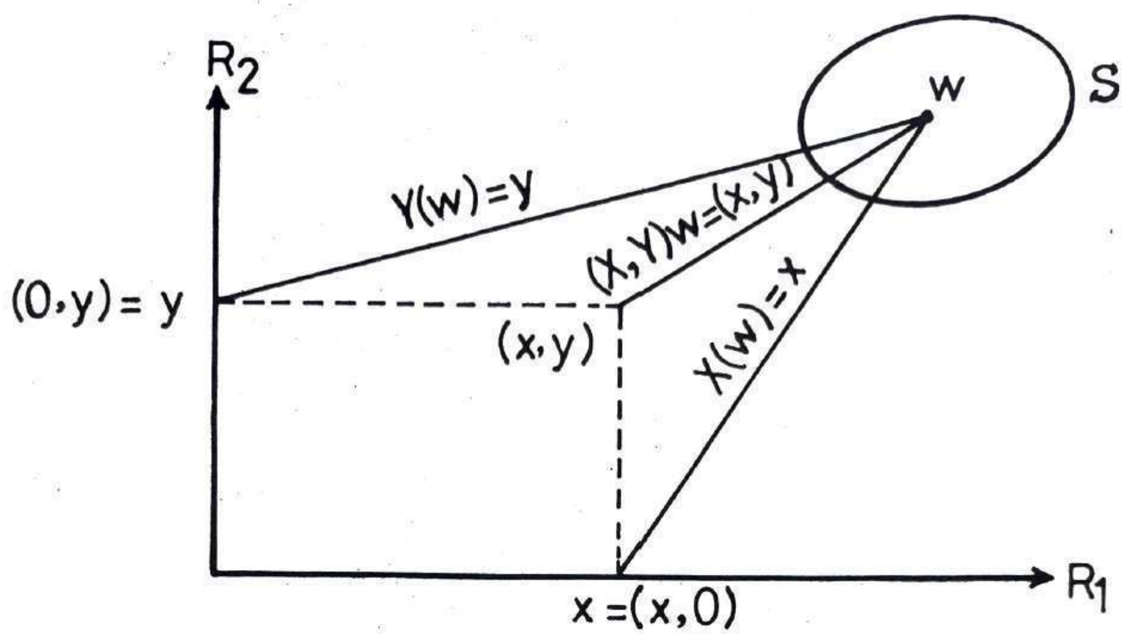


Fig 7.1 Joint mapping of the pair of random variables X and Y.

For one r.v. X ,we define the distribution function as

$$F_X(x) = P\{X \le x\}$$

But the events of the type $\{X \le x, Y \le y\}$ are dependent on the values x and y. The probability function which connects these two events is called the joint distribution function of X and Y, and is denoted by $F_{XY}(x, y)$ or simply F(x,y), and is defined by

$$F(x,y) = P\{X \le x, Y \le y\} \quad \text{.........} \quad (7.1.1)$$

for all real numbers x and y .
Equation (7.1.1) is represented geometrically as in Fig. 7.2.

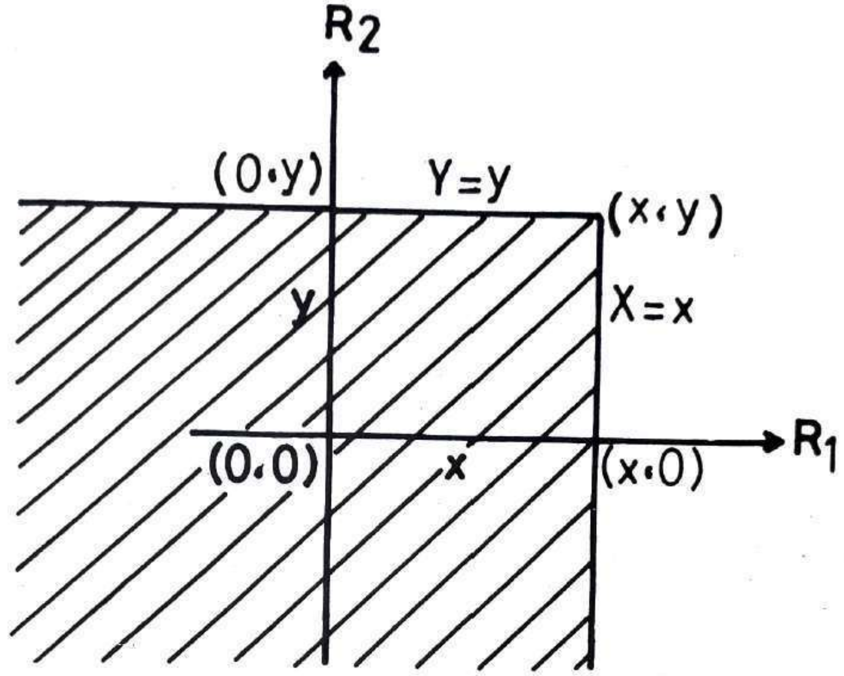


Fig 7.2 Geometiical representation of F(x,y)

The joint distribution function F(x,y) satisfies the following properties:

1· $0 \leq F(x,y) \leq 1, \; -\infty < x, y < \infty$;

2· $\lim_{x \to -\infty} F(x,y) = F(-\infty, y) = 0$;

3· $\lim_{y \to -\infty} F(x,y) = F(x, -\infty) = 0$;

4· $\lim_{\substack{x \to \infty \\ y \to \infty}} F(x,y) = F(\infty, \infty) = 1$;

5· if $x_1 < x_2$ and $y_1 < y_2$ then $F(x_1, y_1) \leq F(x_2, y_2)$.

7.2. Marginal Distributions

Let X and Y be jointly distributed. The distribution function of a r.v. X or Y is called the *marginal distribution function* of X or Y. The marginal distributions of X and Y can be obtained from the *joint probability distribution* F(x,y) as

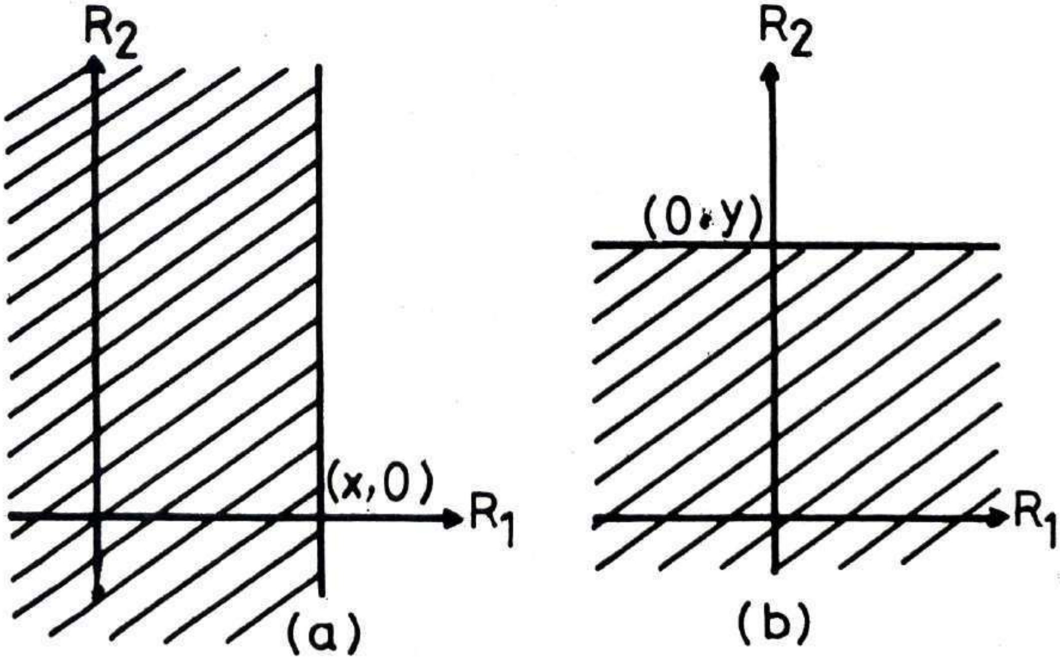
$$F_X(x) = \lim_{y \to \infty} P\{X \leq x, Y \leq y\} = P\{X \leq x\}$$

$$= F(x, \infty) \qquad \text{... (7·2·1)}$$

$$F_Y(y) = \lim_{x \to \infty} P\{X \leq x, Y \leq y\} = P\{Y \leq y\}$$

$$= F(\infty, y) \qquad \text{... (7·2·2)}$$

Where F_X and F_Y are the marginal distribution function of X and Y, respectively. They are represented graphically as in Fig. 7.3



Marginal of x

Marginal of y

Fig. 7.3

7.2.1 Discrete Random Variables

If X and Y are both discrete r.v. 's then the pair (X,Y) is also discrete. The joint probability mass function of X and Y is given by

$$P(X = x_i, Y = y_j) = P_{ij} \quad \ldots (7{\cdot}2{\cdot}3)$$

where i and j take values 1,2,3, ...

we have

$$P_{ij} \geq 0 \text{ for all } i, j$$

and

$$\sum_i \sum_j P_{ij} = 1$$

The mariginal probability mass functions of X and Y can be obtained by

$$P_{i\cdot} = P\{X = x_i\} = P\{X = x_i, Y = y_1\} + P\{X = x_i, Y = y_2\} + \ldots$$
$$= \sum_j P_{ij} \quad \ldots (7{\cdot}2{\cdot}4)$$

and

$$P_{\cdot j} = P\{Y = y_j\} = P\{X = x_1, Y = y_j\} + P\{X = x_2, Y = y_j\} + \ldots$$
$$= \sum_i P_{ij} \quad \ldots (7{\cdot}2{\cdot}5)$$

where $P_{i\cdot}$ and $P_{\cdot j}$ are the marginal p.m.f. of X and Y respectively. Also we have

$$\sum_i P_{i\cdot} = \sum_j P_{\cdot j} = \sum_i \sum_j P_{ij} = 1.$$

and the distribution function F(x,y) is given by

$$F(x, y) = \sum_{x_i \leq x} \sum_{y_j \leq y} P_{ij}.$$

The marginal distribution functions of X and Y can be obtained from

$$F_X(x) = \sum_{x_i \le x} \sum_{j} P_{ij} = \sum_{x_i \le x} P_{i\cdot}$$

and

$$F_Y(y) = \sum_{j} \sum_{y_j \le y} P_{ij} = \sum_{y_j \le y} P_{\cdot j}$$

Example 7.2.1.

Consider an experiment of throwing a fair coin twice. Let X and Y denote the number of heads and tails obtained, respectively. Then the sample space is

S ={HH, HT, TH, TT}

Here

{X=0, Y=2} = {TT}

{X=1, Y=1} = {TH, HT}

{X=2, Y=0} = {HH}

Hence the joint p.m.f. of X,Y is givenb y

$X = x_i$ \ $Y = y_j$	0	1	2	Total $P_{i\cdot}$
0	0	0	$\frac{1}{4}$	$\frac{1}{4}$
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$
2	$\frac{1}{4}$	0	0	$\frac{1}{4}$
Total $P_{\cdot j}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	1

Here P. {X=0, Y=0} =0 because the event {X=0, Y=0} is an impossible event. The same is true for the events {X=0, Y=1}, {X=1, Y=0}, {X=1, Y=2}, {X=2, Y=1} and {X=2, Y=2}.

The marginal p.m.f. 's of X and Y are given by

marginal of X

$X = x_i$	$P(X = x_i)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

marginal of Y

$Y = y_j$	$P(Y = y_j)$
0	$\frac{1}{4}$
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Also we have

$$F(1,2) = \sum_{i \leq 1} \sum_{j \leq 2} P_{ij} = P(X=0, Y=0) + P(X=0, Y=1) +$$

$$P(X=0, Y=2) + P(X=1, Y=1) + P(X=1, Y=2)$$

$$= 0 + 0 + \frac{1}{4} + 0 + \frac{1}{2} + 0 = 3/4.$$

Example 7.2.2

Let X and Y be two r.v.'s with the joint p.m.f. given by

X \ Y	0	1	2	Total P_{x}
−1	$\frac{1}{15}$	$\frac{3}{15}$	$\frac{2}{15}$	6/15
0	$\frac{2}{15}$	$\frac{2}{15}$	$\frac{1}{15}$	5/15
1	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{2}{15}$	4/15
Total P_{y}	$\frac{4}{15}$	$\frac{6}{15}$	$\frac{5}{15}$	1

The marginal p.m.f. X and Y are given by

marginal of X

$X=x_i$	$P(X=x_i)$
-1	6/15
0	5/15
1	4/15

marginal of Y

$Y=y_j$	$P(Y=y_j)$
0	4/15
1	6/15
2	5/15

Also we have

$$F(1,1)=\sum_{x\leq 1,\, y\leq 1}\sum p_{xy}$$

$$=\frac{1}{15}+\frac{3}{15}+\frac{2}{15}+\frac{2}{15}+\frac{1}{15}+\frac{1}{15}$$

$$=10/15.$$

Also

$$F(0,2) = \sum_{i \le 0} \sum_{j \le 2} P_{ij}$$

$$= \frac{1}{15} + \frac{3}{15} + \frac{2}{15} + \frac{2}{15} + \frac{2}{15} + \frac{1}{15}$$

$$= 11/15,$$

and

$$F(1,2) = \sum_{i \le 1} \sum_{j \le 2} P_{ij}$$

$$= 1.$$

7.2.2. Continuous Random Variables

Definition 7.2.1.

The pair of random variables (X,Y) is said to be of a continuous type if there exists a non-negative real valued function $f_{XY}(x,y)$ (or simple f(x,y)), so that for each pair of real numbers (x,y), the joint distribution function of (X,Y) is defined by

$$F(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(x,y)\,dy\,dx \qquad \text{... (7·2·6)}$$

The function f(x,y) is called *the joint density function* of the r.v. 's X and Y.

We have

$$f(x,y) = \frac{\partial^2 F(x,y)}{\partial x\, \partial y}$$

In other words

$$P\{x < X \le x + \Delta x, y < Y \le y + \Delta y\} = f(x,y)\,dxdy\ .$$

The density function f(x,y) satisfies:

i. $f(x,y) \geq 0$ for all x and y

ii. $$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1$$

Fig.7·4 given a geometrical representation of the probability that the point (x,y) lies in the infinitesimal rectangular region of area dxdy.

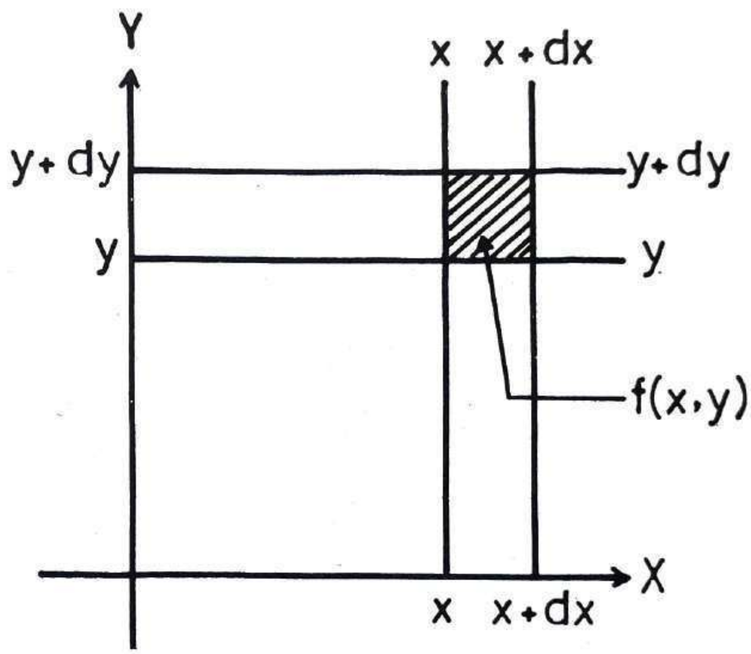


Fig. 7.4

The *marginal distribution function* of X and Y are given by

$$F_X(x) = F(x, \infty) = \int_{-\infty}^{x}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx \quad \ldots (7\cdot2\cdot7)$$

and

$$F_Y(y) = F(\infty, y) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f(x,y)\,dy\,dx \quad \ldots (7\cdot2\cdot8)$$

The marginal density function of X and Y are given by

$$f_X(x) = F_X'(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \quad \ldots\ldots\ldots\ldots (7.2.9)$$

where f_X (x) is the marginal density function of the r.v. X. Similarly, the marginal density function of Y, $f_Y(y)$ is given by

$$f_Y(y) = F'_Y(y) = \int_{-\infty}^{\infty} f(x, y)\,dx \qquad \text{.........} \quad (7.2.10)$$

Example 7.2.3.

Let X and Y be two random variables with joint density function f(x,y) given by.

$$f(x, y) = cxe^{-x(y+1)} \quad x \geq 0, y \geq 0$$

i. Find the constant c.
ii. Find the marginal density functions of X and Y.
iii. Find the marginal distribution functions of X and Y.

Solution

To find the value of c, we use

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\,dy\,dx = 1 \text{, so we have}$$

$$1 = c\int_0^{\infty}\int_0^{\infty} xe^{-x(y+1)}\,dy\,dx = c\int_0^{\infty} e^{-x}\left\{\int_0^{\infty} xe^{-xy}dy\right\}dx$$

$$= c\int_0^{\infty} e^{-x}[-e^{-xy}]_0^{\infty}\,dx$$

$$= c\int_0^{\infty} e^{-x}\,dx = c[-e^{-x}]_0^{\infty} = c$$

$\therefore c = 1$, hence

$f(x, y) = xe^{-x(y+1)}\, x \geq 0, y \geq 0$

To find the marginal density functions of X and Y, we use Equations (7.2.9) and (7.2.10), so we have