LEC5

Mathematical background

- Prime numbers
- Greatest common devisor
- Least Common Multiple (LCM)
- Modular (Modulus) in Mathematics
- Euler’s Totient Function (φ)
- Types of Frequency

2025-2026
Second semester
2026-3-4

Alaa Y. Taqa

Example 1

Find:

$$17 \bmod 5$$

Step 1: Divide 17 by 5

17 ÷ 5 = 3 remainder 2

✅ 17 mod 5 = 2

Example 2

Find:

$$20 \bmod 6$$

20 ÷ 6 = 3 remainder 2

✅ 20 mod 6 = 2

Example 3

Find:

$$45 \bmod 7$$

45 ÷ 7 = 6 remainder 3

✅ 45 mod 7 = 3

Euler's Totient Function (φ)

Definition:

The **Euler's Totient Function**, denoted as **$\varphi(n)$**, is the **number of positive integers less than or equal to n that are coprime with n.**

Two numbers are **coprime** if their **GCD is 1.**

Key Properties

1. $\varphi(1) = 1$
2. If **p** is prime, $\varphi(p) = p - 1$
3. If **p** is prime and $k \geq 1$, $\varphi(p^k) = p^k - p^{(k-1)}$
4. If **m** and **n** are coprime, $\varphi(m \times n) = \varphi(m) \times \varphi(n)$

Greatest Common Divisor (GCD)

The **Greatest Common Divisor (GCD)** is the **largest number that divides two or more numbers exactly (without leaving a remainder).**

Algorithm to Find the Greatest Common Divisor (GCD)

Algorithm (Using the Euclidean Method)

Step 1: Take two positive integers **a** and **b** (assume $a > b$).
Step 2: Divide **a** by **b.**
Step 3: Find the **remainder r.**
Step 4: If **r = 0**, then **b is the GCD.**
Step 5: If **r ≠ 0**, replace **a = b** and **b = r.**
Step 6: Repeat the process from **Step 2** until the remainder becomes **0.**
Step 7: The last non-zero divisor is the **GCD.**

Example 1

Find the GCD of 48 and 18

Step 1: $48 \div 18 = 2$ remainder 12
Step 2: $18 \div 12 = 1$ remainder 6
Step 3: $12 \div 6 = 2$ remainder 0

✅ GCD = 6

Example 2

Find the GCD of 60 and 24

Step 1: $60 \div 24 = 2$ remainder 12
Step 2: $24 \div 12 = 2$ remainder 0

✅ GCD = 12

Example 3

Find the GCD of 81 and 27

Step 1: $81 \div 27 = 3$ remainder 0

✅ GCD = 27

Recursive Algorithm Steps

1. Start with two integers **a** and **b.**
2. If **b = 0**, return **a** (this is the GCD).
3. Otherwise compute the remainder **r = a mod b.**
4. Call the function again with **gcd(b, r).**
5. Repeat until **b becomes 0.**

Example1

Find **GCD(48, 18)**

Step 1:

gcd(48, 18) → remainder = 48 mod 18 = 12

Step 2:

gcd(18, 12) → remainder = 18 mod 12 = 6

Step 3:

gcd(12, 6) → remainder = 12 mod 6 = 0

Step 4:

gcd(6, 0)

✅ **GCD = 6**

Example2

Find **GCD(60, 24)**

gcd(60, 24)

→ gcd(24, 12)

→ gcd(12, 0)

✅ **GCD = 12**

Least Common Multiple (LCM)

The **Least Common Multiple (LCM)** of two or more numbers is the **smallest positive number that is divisible by all the given numbers.**

Algorithm to Find LCM (Using GCD)

Steps of the Algorithm

1. Input two positive integers **a** and **b.**
2. Compute **GCD(a, b)** using the Euclidean algorithm.
3. Multiply the numbers **a × b.**
4. Divide the result by **GCD(a, b)**.
5. The result is the **LCM**.

Example 1

Find LCM of 12 and 18

Step 1: Find GCD

12 = 2 × 2 × 3 ☺

18 = 2 × 3 × 3

Common factors = 2 × 3 = 6

GCD = 6

Step 2: Apply formula

LCM = (12 × 18) / 6

LCM = 216 / 6

LCM = 36

✅ LCM(12,18) = 36

Example 2

Find LCM of 8 and 20

Step 1: Find GCD

8 = 2 × 2 × 2 ☺

20 = 2 × 2 × 5

Common factors = 2 × 2 = 4

GCD = 4

Step 2: Apply formula

LCM = (8 × 20) / 4

LCM = 160 / 4

LCM = 40

✅ LCM(8,20) = 40

Modular (Modulus) in Mathematics

Modular arithmetic deals with the **remainder after division** of one number by another.
It is written using the **mod** symbol.

$$a \bmod b$$

This means the **remainder when** a **is divided by** b.

Algorithm to Find $a \bmod b$

Step 1: Take two integers **a** (dividend) and **b** (divisor).

Step 2: Divide **a** by **b**.

Step 3: Find the **remainder r**.

Step 4: The remainder **r** is the value of $a \bmod b$.

Example 1

Find:

$$17 \bmod 5$$

Step 1: Divide 17 by 5

17 ÷ 5 = 3 remainder 2

✅ **17 mod 5 = 2**

Example 2

Find:

$$20 \bmod 6$$

20 ÷ 6 = 3 remainder 2

✅ **20 mod 6 = 2**

Example 3

Find:

$$45 \bmod 7$$

45 ÷ 7 = 6 remainder 3

✅ **45 mod 7 = 3**

Euler’s Totient Function (φ)

Definition:

The **Euler’s Totient Function**, denoted as **$\varphi(n)$**, is the **number of positive integers less than or equal to n that are coprime with n.**

Two numbers are **coprime** if their **GCD is 1**.

Key Properties

1. $\varphi(1) = 1$
2. If **p** is prime, $\varphi(p) = p - 1$
3. If **p** is prime and $k \geq 1$, $\varphi(p^k) = p^k - p^{(k-1)}$
4. If **m** and **n** are coprime, $\varphi(m \times n) = \varphi(m) \times \varphi(n)$

Algorithm to Find φ(n)

Method 1: Using Factorization

Step 1: Find the prime factorization of n:

$$n = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_m^{k_m}$$

Step 2: Use the formula:

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_m}\right)$$

Example 1

Find φ(12)

Step 1: Prime factorization: $12 = 2^2 \times 3$

Step 2: Apply formula:

$$\phi(12) = 12 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right) = 12 \cdot \frac{1}{2} \cdot \frac{2}{3} = 4$$

✅ **φ(12) = 4**

Numbers coprime with 12: 1, 5, 7, 11

Example 2

Find φ(15)

Step 1: Prime factorization: 15 = 3 × 5

Step 2: Apply formula:

$$\phi(15) = 15 \cdot \left(1 - \frac{1}{3}\right) \cdot \left(1 - \frac{1}{5}\right) = 15 \cdot \frac{2}{3} \cdot \frac{4}{5} = 8$$

✅ **φ(15) = 8**

Numbers coprime with 15: 1, 2, 4, 7, 8, 11, 13, 14

Algorithm (Step-by-Step)

Input: n

Output: φ(n)

1. Initialize φ = n
2. For every prime factor **p** of n:
   - φ = φ × (1 − 1/p)
3. Return φ

Example 3 (φ of a prime number)

φ(17) = 17 − 1 = 16

Numbers coprime with 17: 1, 2, 3, ..., 16

Let's find **Euler's Totient Function φ(5)** step by step.

Step 1: Check if 5 is prime

Yes, **5 is a prime number.**

Step 2: Apply the prime formula

If **p is prime**, then:

$$\phi(p) = p - 1$$

So:

$$\phi(5) = 5 - 1 = 4$$

Step 3: Verify by counting numbers coprime with 5

Numbers **less than 5:** 1, 2, 3, 4
Check which are coprime with 5 (GCD = 1 with 5):

- gcd(1,5) = 1 → coprime
- gcd(2,5) = 1 → coprime
- gcd(3,5) = 1 → coprime
- gcd(4,5) = 1 → coprime

✅ Total = 4 numbers → matches φ(5) = 4

Let's calculate **Euler's Totient Function φ(6)** step by step.

Step 1: Find prime factorization of 6

6 = 2 × 3

Step 2: Apply Euler's formula

$$\phi(n) = n \cdot \left(1 - \frac{1}{p_1}\right) \cdot \left(1 - \frac{1}{p_2}\right) \ldots$$

Here, $n = 6$, $p_1 = 2$, $p_2 = 3$:

$$\phi(6) = 6 \cdot \left(1 - \frac{1}{2}\right) \cdot \left(1 - \frac{1}{3}\right)$$

$$\phi(6) = 6 \cdot \frac{1}{2} \cdot \frac{2}{3} = 2$$

Step 3: Verify by counting numbers coprime with 6

Numbers less than 6: 1, 2, 3, 4, 5

Check GCD with 6:

- gcd(1,6) = 1 → coprime
- gcd(2,6) = 2 → not coprime
- gcd(3,6) = 3 → not coprime
- gcd(4,6) = 2 → not coprime
- gcd(5,6) = 1 → coprime

✅ Coprime numbers: 1, 5 → total 2 → matches φ(6) = 2

So the **Euler function of 6 is 2**.

Quick Rule

- If **GCD(a, b) = 1**, then **a and b are coprime.**
- If GCD > 1 → not coprime.

Coprime numbers don’t have to be prime themselves!

- Example: 8 and 15 are **coprime**, even though neither is prime.

Frequency

In **statistics, frequency** is the **number of times a particular value or category occurs in a dataset.**
It helps summarize data and see patterns quickly.

Example:
Dataset = [2, 3, 3, 5, 2, 3, 5, 2]

Value	**Frequency**
2	3
3	3
5	2

Types of Frequency

1. Absolute Frequency (f)
    1. The **actual count** of occurrences of a value.
    2. Example: In [2, 3, 3, 5, 2], the absolute frequency of 2 is **2.**
2. Relative Frequency (rf)
    1. The **proportion of the total observations.**
    2. Formula:

$$\text{Relative } Frequency = \frac{Frequency\ of\ a\ value}{Total\ number\ of\ observations}$$

    1. Example: For 2 in [2, 3, 3, 5, 2], total = 5

$$\text{rf} = \frac{2}{5} = 0.4$$

1. Cumulative Frequency (CF)
    1. The **sum of frequencies** up to a certain value in a dataset.
    2. Helps in finding percentiles and medians.

Example Dataset

Data: [2, 3, 3, 5, 2, 3, 5, 2]

Step 1: Find Absolute Frequency

Value	Absolute Frequency (f)
2	3
3	3
5	2

Step 2: Find Relative Frequency

$$\mathrm{rf} = \frac{f}{\text{Total observations}} = \frac{f}{8}$$

Value	f	rf
2	3	3/8 = 0.375
3	3	3/8 = 0.375
5	2	2/8 = 0.25

Step 3: Find Cumulative Frequency (CF)

Value	f	CF
2	3	3
3	3	6
5	2	8

✅ **Cumulative frequency** = total observations counted up to that value.


Questions


THANK
YOU