University of Mosul

College of Computer Science and Mathematics
Department of Statistics and Informatics
Stochastic Processes (II)
Level 4

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Lecture 1: Classification of Markov Chain

(1-1) Introduction:

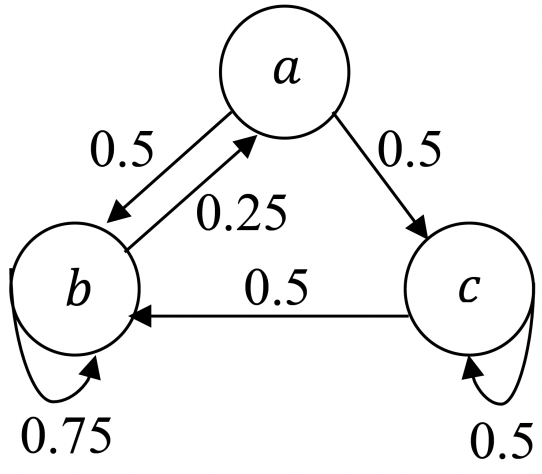
In this section the topic was presented a classification of states and the chain of a Markov chain.

(1-2) Transition Diagram:

The transition probability of Markov Chain can be represented by a diagram call (Transition Diagram), where a probability p_{ij} is denoted by an arrow from state i to state j, for example consider the following transition matrix:

$$P = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.25 & 0.75 & 0 \\ 0 & 0.5 & 0.5 \end{bmatrix}, \quad S = \{a, b, c\}$$

Then the transition diagram for this process is:



(1-3) Classification of Chains:

1) Accessibility:

If the state j of chain can be reached from state i in any number of transitions, i.e:

$$p_{ij}^{(n)} > 0,$$

for some $n \geq 0$, then we said that the state j is accessibility from state i, and it write by $(i \to j)$.

2) Irreducible chain:

If every state can be reached from every other state (in any number of transition), then the chain is said to be irreducible, and the transition matrix is irreducible.

3) Communication states:

Two states i and j each accessible to each other, then they are said to be communication states, and it write by $(i \leftrightarrow j)$. The communication has the following properties:

1. Reflexive: for any i and j, then $(i \leftrightarrow j)$.
2. Symmetric: if $(i \leftrightarrow j)$, then $(j \leftrightarrow i)$.
3. Transitive: if $(i \leftrightarrow j)$ and $(j \leftrightarrow k)$, then $(i \leftrightarrow k)$.

4) Closed set of states:

If C is a set of states such that no state outside of C can be reached from any state in C, then the set C is said to be closed. If the set C is closed set and $j \in C$ while $k \notin C$, then $p_{jk} = 0$. It can be seen that:

$$p_{jk}^{(2)} = 0.$$

In general:

$$p_{jk}^{(n)} = 0 \quad \forall n \geq 1.$$

5) Absorbing states:

If a closed set C contain only one state j, then the state j is called an absorbing state, if and only if:

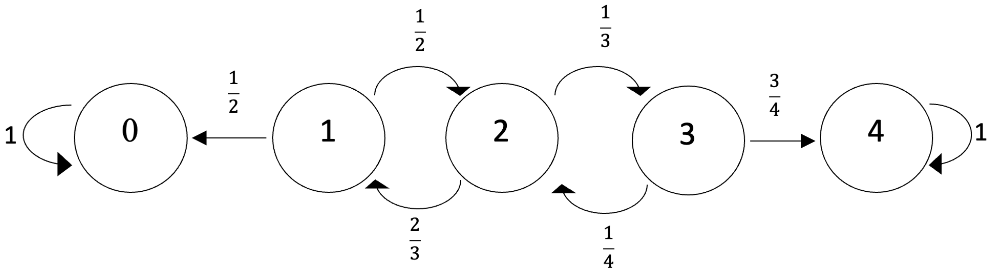
$$p_{jj} = 1 \quad \text{and} \quad p_{jk} = 0 \ \forall \ k \neq j.$$

Remarks:

1. The set of all states of Markov chain will be a closed set.
2. If a Markov chain does not contain any other closed set, then the chain will be irreducible.
3. The chains which are not irreducible are called reducible, in this case the number of closed sets is two or more than one state.

Example (1.1):

Classified this Markov chain with transition diagram:



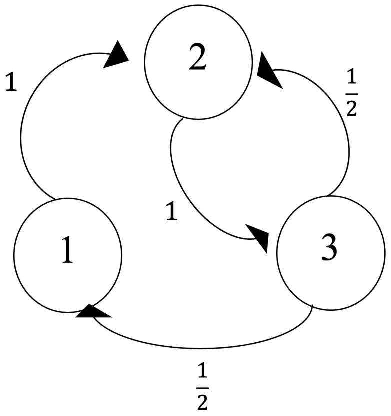
1. **Closed sets are:** $C_1 = \{0\}$, and $C_2 = \{4\}$.
2. Not closed sets is the set $C_3 = \{1, 2, 3\}$ since the chain can leave it to 0 (from 1) and to 4 (from 3).
3. **Reducibility:** Since there is more than one closed communicating class (namely $\{0\}$ and $\{4\}$), the chain is *reducible.*
4. **Absorbing states:** Since the closed sets $\{0\}$ and $\{4\}$ consist of a single state, the states 0 and 4 are absorbing (equivalently, $p_{00} = 1$ and $p_{44} = 1$).
5. **Accessibility:** The accessibility relation is determined as follows:
    - For any $i, j \in \{1, 2, 3\}$, we have $i \to j$ (there exists $n \geq 1$ such that $p_{ij}^{(n)} > 0$).
    - From $\{1, 2, 3\}$, the chain can reach 0 and 4 (since $1 \to 0$ and $3 \to 4$), hence $i \to 0$ and $i \to 4$ for all $i \in \{1, 2, 3\}$.
    - From 0, the chain cannot reach any other state, so $0 \nrightarrow j$ for all $j \neq 0$.
    - From 4, the chain cannot reach any other state, so $4 \nrightarrow j$ for all $j \neq 4$.
6. **Communication classes:** The communicating (equivalence) classes are $C_1 = \{0\}$, $C_2 = \{4\}$, and $C_3 = \{1, 2, 3\}$. In particular, states $\{1, 2, 3\}$ communicate with each other ($i \leftrightarrow j$), while 0 and 4 each form a singleton communicating class.

Example (1.2):

If we have a Markov chain with state space $\{1, 2, 3\}$ and transition matrix P, then classify the chain:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution:



1. Closed set is: $C = \{1, 2, 3\}$
2. Since there is one closed set, then the chain is irreducible.
3. No absorbing state because $p_{ii} \neq 1$ and $p_{jk} > 0$ for some $k \neq j$.
4. States $\{1, 2, 3\}$ are accessibility because $p_{ij}^{(n)} > 0$, for some $n \geq 0$.
5. States $\{2, 3\}$ are communication states because they can accessible each to other $(i \leftrightarrow j)$.

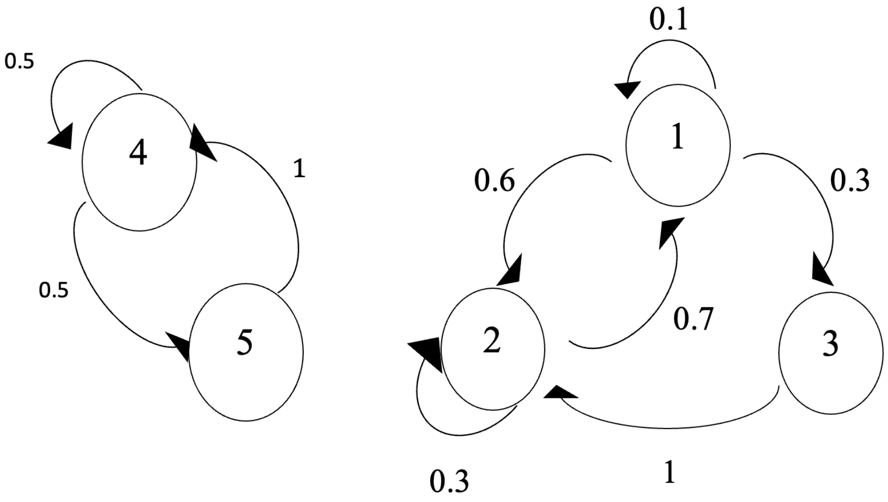
Example (1.3):

Classify the following Markov chain with transition matrix and state space:

$$S = \{1, 2, 3, 4, 5\}$$

$$P = \begin{bmatrix} 0.1 & 0.6 & 0.3 & 0 & 0 \\ 0.7 & 0.3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution:



there are no transitions from states to states

1. From the matrix P,we have $C_1 = \{1, 2, 3\}$ and $C_2 = \{4, 5\}$. Since no transitions from states C_1 to C_2 and vice versa, then C_1 and C_2 are closed classes.
2. Since the state space contains more than one closed communicating class, the Markov chain is *reducible.*
3. We find that $p_{ii} \neq 1$ and $p_{jk} > 0$ for some $k \neq j.$. Hence, the chain has no absorbing states.
4. States $\{1, 2\}$ and $\{4, 5\}$ are communication states because they can accessible each to other $(i \leftrightarrow j)$.