

Lecture 2: Classification of State of Markov Chains

Definition of the First Passage:

Suppose that a system starts with state j , and let $f_{jk}^{(n)}$ be the probability that it reaches the state k for the first time at n^{th} step (or after n transition), and let $p_{jk}^{(n)}$ be the probability that it reaches the state k (not necessarily for the first time) after n^{th} steps, then the probability of the first passage is:

$$f_{jk}^{(n)} = Pr\{X_n = k, X_r = i; r = 1, 2, \dots, n-1 \mid X_0 = j\}$$

and:

$$p_{jj}^{(n)} = Pr\{X_n = j, X_r = i; r = 1, 2, \dots, n-1 \mid X_0 = j\}$$

Let F_{jk} denote the probability that the system starting with state j , the system will ever reach the state k as:

$$F_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)}$$

Then we have two cases:

$$F_{jk} = 1 \quad \text{or} \quad F_{jk} < 1$$

If $F_{jk} = 1$, this means that the system is in state j and certainly it will reach to state k .

And if $F_{jk} < 1$, this means that the system is in state j and uncertainty it will reach to state k .

Then the mean of the first passage time from state j to state k is given by:

$$\mu_{jk} = \sum_{n=1}^{\infty} n f_{jk}^{(n)},$$

where $\{f_{jk}^{(n)}, n = 1, 2, \dots\}$ the probability of the first passage.

Classification of states of Markov chain:

1. **Recurrent and Transient states:** If $j = k$, then F_{jj} will represent the distribution of the recurrent times of state j , then:

- If $F_{jj} = 1$, then the state j is said to be a recurrent (i.e., return to state j is certain).
- If $F_{jj} < 1$, then the state j is said to be a transient (i.e., return to state j is uncertain).

2. **Positive and Null recurrent states:**

- A recurrent state is said to be Positive recurrent if the mean recurrent μ_{jj} for the state j is finite (i.e., $\mu_{jj} < \infty$).

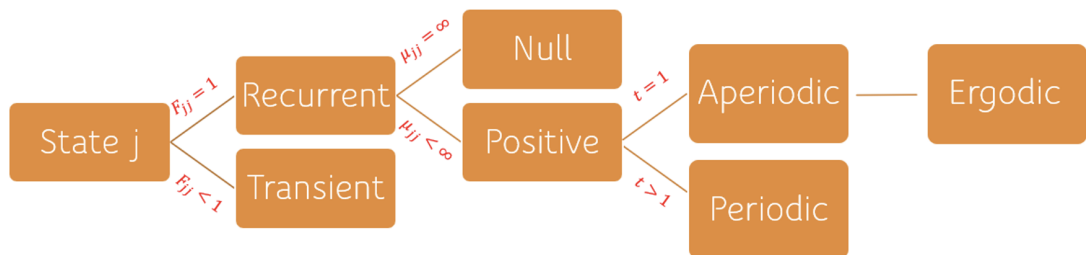
- A recurrent state is said to be Null recurrent if the mean recurrent μ_{jj} for the state is infinite (i.e., $\mu_{jj} = \infty$).

3. Periodic and Aperiodic states:

- A state j is said to be periodic with period t ($t > 1$) if return to the state is possible only at $(t, 2t, 3t, \dots)$ steps, when t is greatest integer with this property.
- A state j is said to be aperiodic (or non-periodic) if no such $(t \geq 1)$ exists, for example ($t = 1$).

4. **Ergodic:** A recurrent positive and aperiodic state of the Markov chain is said to be ergodic. A Markov chain with the states ergodic is said to be ergodic chain.

Classifications of the states of Markov Chain

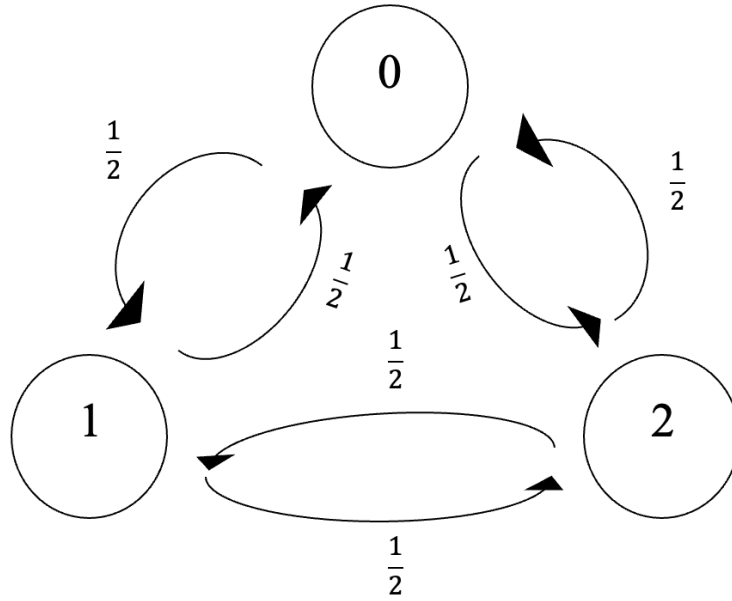


Example:

If we have a Markov chain with state space $\{0, 1, 2\}$ and transition matrix, classify the states of this Markov chain:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Solution:



State (0):

1. Probability distribution of the first passage for the state is:

$$F_{00} = \sum_{n=1}^{\infty} f_{00}^{(n)}$$

$$f_{00}^{(1)} = 0$$

$$f_{00}^{(2)} = \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} = \frac{1}{2}$$

$$f_{00}^{(3)} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$$

$$f_{00}^{(4)} = \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$$

$$F_{00} = f_{00}^{(1)} + f_{00}^{(2)} + f_{00}^{(3)} + f_{00}^{(4)} + \dots$$

$$F_{00} = 0 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

$$= \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1$$

Since the prob. dist. of the first passage of state (0) is ($F_{00} = 1$), then the state (0) is **recurrent state**.

2. The mean recurrent time of the state (0) is:

$$\begin{aligned}
\mu_{00} &= \sum_{n=1}^{\infty} n f_{00}^{(n)} \\
&= 1(0) + 2 \left(\frac{1}{2}\right) + 3 \left(\frac{1}{2}\right)^2 + 4 \left(\frac{1}{2}\right)^3 + \dots \\
&= 0 + 2 \left(\frac{1}{2}\right) + 3 \left(\frac{1}{2}\right)^2 + 4 \left(\frac{1}{2}\right)^3 + \dots \\
&= \sum_{n=2}^{\infty} n \left(\frac{1}{2}\right)^{n-1} = 2 \sum_{n=2}^{\infty} n \left(\frac{1}{2}\right)^n + 1 - 1 \\
&= 2 \left[\sum_{n=2}^{\infty} n \left(\frac{1}{2}\right)^n + \frac{1}{2} - \frac{1}{2} \right] \\
&= 2 \left[\sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n - \frac{1}{2} \right] \\
&= 2 \sum_{n=1}^{\infty} n \left(\frac{1}{2}\right)^n - 1 = 2 \left(\frac{\frac{1}{2}}{\left(1 - \frac{1}{2}\right)^2} \right) - 1 \\
&= 2 \left(\frac{\frac{1}{2}}{\left(\frac{1}{2}\right)^2} \right) - 1 = 3 < \infty
\end{aligned}$$

Then the state (0) is **positive recurrent**.

3. $(0) \rightarrow (1) \rightarrow (0), \quad t = 2$
 $(0) \rightarrow (1) \rightarrow (2) \rightarrow (0), \quad t = 3$
 $(0) \rightarrow (1) \rightarrow (2) \rightarrow (1) \rightarrow (0), \quad t = 4$

Since it returns to state (0) at $(t = 2, 3, 4, \dots)$ steps, then the state (0) is **aperiodic**.

4. Since the state (0) is positive recurrent with aperiodic state, then the state (0) is **ergodic**.

And also, state (1) and state (2).

Since all states of M.C. are ergodic then the chain is **ergodic chain**.