Lecture 4: Stationary Distribution of Markov Chain

Definition:

Consider an irreducible positive recurrent and aperiodic M.C. (i.e., ergodic chain), then the probability distribution $\{\pi_j\}$ is called the stationary distribution of this chain if the system of this linear equation:

$$\pi_j = \sum_{j \in S} \pi_j p_{jj}, \quad j \in S$$

where:

$$\sum_{j \in S} \pi_j = 1 \quad \text{and} \quad \pi_j \geq 0$$

has a solution: $\Pi = [\pi_1 \ \pi_2 \ \cdots]$
is exist solution and there is one solution, and:

$$\pi_j = \lim_{n \to \infty} p_{jj}^{(n)}$$

Theorem (2):

The stationary distribution Π is verify the equation:

$$\Pi(I - P) = 0$$

where I is an identity matrix.

Proof:

From Chapman-Kolmogorov equation, we have:

$$P^{(n)} = P \cdot P^{(n-1)}$$

Since:

$$\Pi = \lim_{n \to \infty} P^{(n)}, \quad \text{then:}$$

$$\Pi = \lim_{n \to \infty} P^{(n-1)} P$$

$$\Pi = \Pi \cdot P \quad \Rightarrow \quad \Pi - \Pi \cdot P = 0$$

$$\Pi(I - P) = 0$$

this is proof of theorem.

Example:

Let a Markov chain with state space $\{0, 1, 2\}$ and transition matrix is:

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

If the chain is ergodic, find the stationary distribution.

Solution:

Since the stationary distribution is verify the equation:

$$\Pi(I - P) = 0$$

then:

$$[\pi_1 \ \pi_2 \ \pi_3] \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix} \right) = [0 \ 0 \ 0]$$

$$[\pi_1 \ \pi_2 \ \pi_3] \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} = [0 \ 0 \ 0]$$

$$\pi_1 - \frac{1}{2}\pi_2 - \frac{1}{2}\pi_3 = 0 \qquad (1)$$

$$-\frac{1}{2}\pi_1 + \pi_2 - \frac{1}{2}\pi_3 = 0 \qquad (2)$$

$$-\frac{1}{2}\pi_1 - \frac{1}{2}\pi_2 + \pi_3 = 0 \qquad (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$$

Multiply equation (1), (2) and (3) by 2, we have:

$$2\pi_1 - \pi_2 - \pi_3 = 0 \qquad (1)$$

$$-\pi_1 + 2\pi_2 - \pi_3 = 0 \qquad (2)$$

$$-\pi_1 - \pi_2 + 2\pi_3 = 0 \qquad (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$$

Subtract eq.(3) from eq.(2), we have:

$$3\pi_2 - 3\pi_3 = 0 \Rightarrow \pi_2 - \pi_3 = 0$$

$$\Rightarrow \pi_2 = \pi_3 \qquad (5)$$

Put (5) in (1), we have:

$$2\pi_1 - \pi_2 - \pi_2 = 0 \Rightarrow 2\pi_1 - 2\pi_2 = 0$$

$$\Rightarrow \pi_1 = \pi_2 \qquad (6)$$

Then:

$$\pi_1 = \pi_2 = \pi_3 = \frac{1}{3}$$

$$\therefore\ \Pi = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

the stationary distribution.

Example:

Find the stationary distribution for this ergodic Markov chain with state space $\{1, 2, 3\}$ and transition matrix:

$$P = \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0 & 0.4 \\ 0 & 0.4 & 0.6 \end{bmatrix}$$

Solution:

Since:

$$\Pi(I - P) = 0$$

then:

$$[\pi_1\ \pi_2\ \pi_3] \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.3 & 0.5 & 0.2 \\ 0.6 & 0 & 0.4 \\ 0 & 0.4 & 0.6 \end{bmatrix} \right) = [0\ 0\ 0]$$

$$[\pi_1\ \pi_2\ \pi_3] \begin{bmatrix} 0.7 & -0.5 & -0.2 \\ -0.6 & 1 & -0.4 \\ 0 & -0.4 & 0.4 \end{bmatrix} = [0\ 0\ 0]$$

$$0.7\pi_1 - 0.6\pi_2 = 0 \qquad (1)$$

$$-0.5\pi_1 + \pi_2 - 0.4\pi_3 = 0 \qquad (2)$$

$$-0.2\pi_1 - 0.4\pi_2 + 0.4\pi_3 = 0 \qquad (3)$$

$$\pi_1 + \pi_2 + \pi_3 = 1 \qquad (4)$$

From eq.(1), we have:

$$\pi_1 = \frac{0.6}{0.7}\pi_2 \qquad (5)$$

Put (5) in (2), we have:

$$-0.5\left(\frac{0.6}{0.7}\right)\pi_2 + \pi_2 - 0.4\pi_3 = 0$$

$$-0.42\pi_2 + \pi_2 - 0.4\pi_3 = 0$$

$$\pi_3 = \frac{0.58}{0.4}\pi_2 \qquad (6)$$

Put (5) and (6) in (4), we have:

$$\frac{0.6}{0.7}\pi_2 + \pi_2 + \frac{0.58}{0.4}\pi_2 = 1$$

$$\therefore \ \pi_2 = 0.3$$

then:

$$\pi_1 = \frac{0.6}{0.7}(0.3) = 0.26$$

and:

$$\pi_3 = \frac{0.58}{0.4}(0.3) = 0.44$$

Then:

$$\Pi = [0.26 \ \ 0.3 \ \ 0.44]$$

the stationary distribution.

Remark:

The relation between the stationary distribution of the states and the mean recurrent time is:

$$\mu_j = \frac{1}{\pi_j} \quad \text{or} \quad \pi_j = \frac{1}{\mu_j}$$

This is means that:

1. If $\mu_{jj} = \infty$ then the recurrent state is said to be null recurrent;

$$\pi_j = \frac{1}{\infty} = 0 \quad \text{(non-stationary)}$$

2. If $\mu_{jj} < \infty$ then the recurrent state is said to be positive recurrent;

$$\pi_j = \frac{1}{\mu_{jj}} > 0 \quad \text{(stationary)}$$