# Lecture 6: Examples of the Poisson process

## Example:

The number of customers arriving at a market can be modeled by a Poisson process with intensity $\lambda = 12$ customers per hour.

1. Find the probability that there are 2 customers between 10:00 and 10:20.
2. Find the probability that there are 3 customers between 10:00 and 10:20 and 7 customers between 10:20 and 11.

**Solution:**

1. Here, $\lambda = 12$ and the interval between 10:00 and 10:20 has length $t = 20$ minutes. Thus, if $X$ is the number of arrivals in that interval, we can write

$$N_t \sim Poisson\left(\frac{12}{60}\right) \text{ per minutes.}$$

Therefore:

$$Pr\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \qquad n = 0, 1, 2, \dots$$

$$Pr\{N(20) = 2\} = \frac{e^{-\frac{12}{60}(20)}\left(\frac{12}{60}(20)\right)^2}{2!} = 8e^{-4} = 0.147$$

2. We have two non-overlapping intervals:

$$I_1 = (10{:}00\ a.m.,\ 10{:}20\ a.m.]$$

$$I_2 = (10{:}20\ a.m.,\ 11\ a.m.]$$

Thus, we can write:

$P(3\, arrivals\, in\, I_1\, \&\, 7\, arrivals\, in\, I_2) = P(3\, arrivals\, in\, I_1)\, P(7\, arrivals\, in\, I_2)$

(Independent increment), then:

$$Pr\{N(20) = 3\} \cdot Pr\{N(40) = 7\} = e^{-\frac{12}{60}(20)}\frac{\left(\frac{12}{60}(20)\right)^3}{3!} \cdot e^{-\frac{12}{60}(40)}\frac{\left(\frac{12}{60}(40)\right)^7}{7!}$$

$$= 4438.42\, e^{-4}e^{-8} = 4438.42\, e^{-12} = 0.027$$

**Example:**

Defects occur along an undersea cable according to a Poisson process of rate $\lambda = 0.1$ per mile.

1. What is the probability that no defects appear in the first two miles of cable?
2. Given that there are no defects in the first two miles of cable, what is the conditional probability of no defects between mile points two and three?

To answer (1) we observe that $N(2)$ has a Poisson distribution whose parameter is $(0.1)(2) = 0.2$. Thus:

$$Pr\{N(2) = 0\} = \frac{e^{-0.2}(0.2)^0}{0!} = e^{-0.2} = 0.8187$$

In part (2) we use the independence of $N(3) - N(2)$ and $N(2) - N(0) = N(2)$. Thus, the conditional probability is the same as the unconditional probability, and:

$$Pr\{N(3) - N(2) = 0\} = Pr\{N(3-2) = 0\} = Pr\{N(1) = 0\}$$

$$= \frac{e^{-0.1(1)}(0.1(1))^0}{0!} = e^{-0.1} = 0.9048$$

**Example:**

Customers arrive in a certain store according to a Poisson process of rate $\lambda = 4$ per hour. Given that the store opens at 9:00 A.M., what is the probability that exactly one customer has arrived by 9:30 and a total of five have arrived by 11:30 A.M.?

Measuring time $t$ in hours from 9:00 A.M., we are asked to determine:

$$Pr\left\{N\left(\frac{1}{2}\right) = 1,\ N\left(\frac{5}{2}\right) = 5\right\}$$

We use the independence of $N\left(\frac{5}{2}\right) - N\left(\frac{1}{2}\right)$ and $N\left(\frac{1}{2}\right)$ to reformulate the question thus:

$$Pr\left\{N\left(\frac{1}{2}\right) = 1,\ N\left(\frac{5}{2}\right) = 5\right\} = Pr\left\{N\left(\frac{1}{2}\right) = 1,\ N\left(\frac{5}{2} - \frac{1}{2}\right) = 5 - 1\right\}$$

$$= Pr\left\{N\left(\frac{1}{2}\right) = 1\right\} \cdot Pr\{N(2) = 4\}$$

$$= \left\{\frac{e^{-4\left(\frac{1}{2}\right)}\left(4\left(\frac{1}{2}\right)\right)^1}{1!}\right\} \cdot \left\{\frac{e^{-4(2)}\left((4)(2)\right)^4}{4!}\right\} = 0.0155$$

**Example:**

Suppose that the process $N(t)$ has a Poisson process with rate $\lambda = 8$, then the probability:

$$Pr\{N(2.5) = 17,\ N(3.7) = 22,\ N(4.3) = 36\}$$

$$= Pr\{N(2.5) = 17,\ N(3.7 - 2.5) = 22 - 17,\ N(4.3 - 3.7) = 36 - 22\}$$

$$= Pr\{N(2.5) = 17\} \cdot Pr\{N(1.2) = 5\} \cdot Pr\{N(0.6) = 14\}$$

$$= \left\{ \frac{e^{-(8)2.5} \left((8)2.5\right)^{17}}{17!} \right\} \cdot \left\{ \frac{e^{-(8)1.2} \left((8)1.2\right)^{5}}{5!} \right\} \cdot \left\{ \frac{e^{-(8)0.6} \left((8)0.6\right)^{14}}{14!} \right\} = 0.000001$$