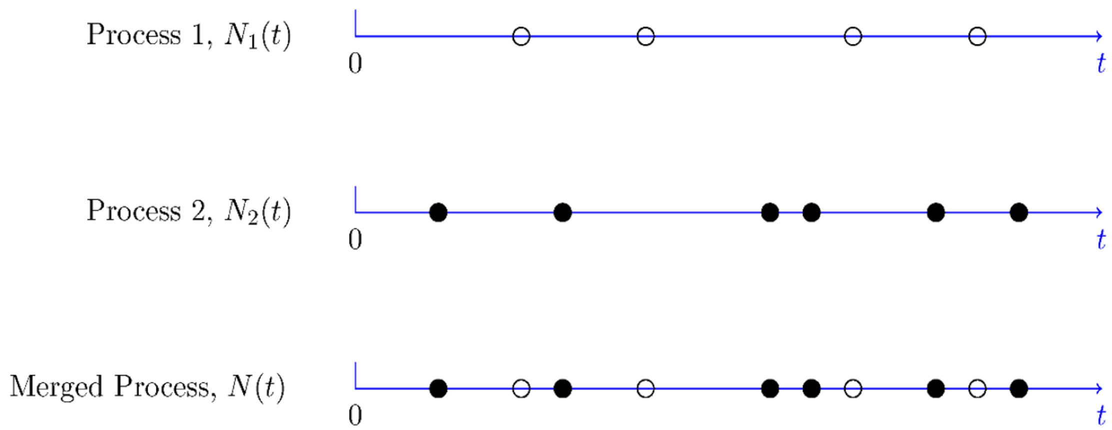
Lecture 7: Properties of Poisson Process

Merging Independent Poisson Processes:

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Let us define:

$$N(t) = N_1(t) + N_2(t)$$

That is, the random process $N(t)$ is obtained by combining the arrivals in $N_1(t)$ and $N_2(t)$. In other words, $\{N(t), t \geq 0\}$ is the process consisting of all arrivals to both process 1 and process 2. We claim that $N(t)$ is a Poisson process with rate $\lambda = \lambda_1 + \lambda_2$.



Since $N_1(t)$ and $N_2(t)$ are independent and both have independent increments, we conclude that $N(t)$ also has independent increments.

Theorem:

Let $N_1(t), N_2(t), \ldots, N_m(t)$ be m independent Poisson processes with rates $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively. Let also:

$$N(t) = N_1(t) + N_2(t) + \cdots + N_m(t), \quad \text{for all } t \in (0, \infty].$$

Then $N(t)$ is a Poisson process with rate $(\lambda_1 + \lambda_2 + \cdots + \lambda_m)$.

Example:

Suppose that the cars arrive to station from three ways independent Poisson process with arrival rate $(2, 4, 6)$ respectively. Find the probability of there is less than two cars arrive to station in 3 minutes.

Solution:

From the properties of Poisson process, we have:

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 2 + 4 + 6 = 12, \qquad t = 3$$

$$\text{The mean } = \lambda t = 12(3) = 36$$

$$Pr\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$Pr\{N(3) < 2\} = Pr\{N(3) = 1\} + Pr\{N(3) = 0\}$$

$$= \frac{e^{-36}(36)^1}{1!} + \frac{e^{-36}(36)^0}{0!} = 37e^{-36}$$

Difference of two independent Poisson Processes:

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates λ_1 and λ_2 respectively. Let define $N(t) = N_1(t) - N_2(t)$. Then the random process $N(t)$ has a distribution given by:

$$Pr\{N(t) = n\} = e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} I_n\left(2t\sqrt{\lambda_1\lambda_2}\right)$$

When $n = 0, \pm 1, \pm 2, \ldots$ and:

$$I_n(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2r+n}}{r!\Gamma(r+n+1)}$$

is the modified Bessel function of order $n \geq -1$.

Proof:

$$Pr\{N(t) = n\} = \sum_{r=0}^{\infty} Pr\{N_1(t) = n + r\}\, Pr\{N_2(t) = r\}$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda_1 t}(\lambda_1 t)^{n+r}}{(n+r)!}\frac{e^{-\lambda_2 t}(\lambda_2 t)^r}{r!}$$

$$= e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} \sum_{r=0}^{\infty} \frac{\left(t\sqrt{\lambda_1\lambda_2}\right)^{2r+n}}{r!(r+n)!}$$

$$= e^{-(\lambda_1+\lambda_2)t}\left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{n}{2}} I_n\left(2t\sqrt{\lambda_1\lambda_2}\right)$$

Where:

$$I_n(x) = \sum_{r=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2r+n}}{r!\Gamma(r+n+1)}$$

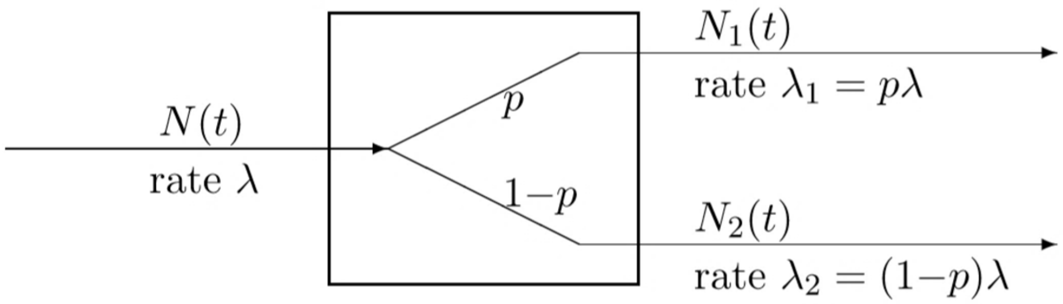
Thus, the difference of two independent Poisson process is not Poisson process.

Splitting of Poisson Processes:

Let $N(t)$ be a Poisson process with rate λ. Here, we divide $N(t)$ to two processes $N_1(t)$ and $N_2(t)$ in the following way (Figure blew).

For each arrival, a coin with $P(H) = p$ is tossed. If the coin lands heads up, the arrival is sent to the first process $N_1(t)$, otherwise it is sent to the second process. The coin tosses are independent of each other and are independent of $N(t)$. Then,

1. $N_1(t)$ is a Poisson process with rate λp.
2. $N_2(t)$ is a Poisson process with rate $\lambda(1-p)$.
3. $N_1(t)$ and $N_2(t)$ are independent.



Each arrival is independently sent to process 1 with probability p and to process 2 otherwise.

Example:

Suppose that the sale of computer in a shop is follows Poisson process with rate $\lambda = 7$ computer per day, and if the number of type laptop computers sold 3 of 10 computers per day, find the probability that:

1. There are four laptops sold in two days.
2. Sale three computers in two days not of type laptop.
3. Sale five computers per day.

Solution:

1. Let $N_1(t)$ sale laptop in shop with Poisson process with probability:

$$p = \frac{3}{10} = 0.3$$

then the rate of sale laptop is:

$$\lambda_1 = \lambda p = 7(0.3) = 2.1$$

$$Pr\{N_1(t) = n\} = \frac{e^{-\lambda_1 t}(\lambda_1 t)^n}{n!}$$

$$Pr\{N_1(2) = 4\} = \frac{e^{-(2.1)(2)}((2.1)(2))^4}{4!} = 12.97e^{-4.2}$$

2. Let $N_2(t)$ sale computers of type not laptop in shop with Poisson process with probability:

$$1 - p = 1 - 0.3 = 0.7$$

Then the rate sale computers of type not laptop is:

$$\lambda_2 = \lambda(1 - p) = 7(0.7) = 4.9$$

$$Pr\{N_2(t) = n\} = Pr\{N_2(2) = 3\} = \frac{e^{-(4.9)(2)}((4.9)(2))^3}{3!} = 156.8e^{-9.8}$$

3. Sale five computers from shop, then:

$$n = 5, \quad t = 1, \quad \lambda = 7$$

$$Pr\{N(t) = n\} = Pr\{N(1) = 5\} = \frac{e^{-7(1)}(7(1))^5}{5!} = 140.05e^{-7}$$