Lecture 8: Solved Problems

Problem:

Suppose that patients arriving at a hospital through three cities follow a Poisson process with arrival rates $(1, 3, 5)$ per minute respectively. Find the probability of more than or equal two patients arriving at the hospital in 3 minutes.

Solution

From the properties of the Poisson process, we have:

$$\lambda = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 3 + 5 = 9, \qquad t = 3$$

$$\text{The mean } = \lambda t = 9(3) = 27$$

$$Pr\{N(t) = n\} = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$Pr\{N(3) \geq 2\} = 1 - Pr\{N(3) < 2\}$$

$$= 1 - [Pr\{N(3) = 1\} + Pr\{N(3) = 0\}]$$

$$= 1 - \left[\frac{e^{-27}(27)^1}{1!} + \frac{e^{-27}(27)^0}{0!}\right]$$

$$= 1 - 28e^{-27}$$

Problem :

A call center receives an average of 10 calls per hour. Let $N(t)$ be the number of calls that have arrived up to time t. Determine the following probabilities and conditional probabilities:

1. $P(N(1) = 3)$, i.e., the probability that exactly 3 calls arrive in the first hour.
2. $P(N(3) > 2)$, i.e., the probability that more than 2 calls arrive in the first three hours.
3. $P(N(2) = 3 \mid N(1) = 1)$, i.e., the probability that exactly 3 calls arrive in the second hour given that exactly one call arrived in the first hour.

Solution

1.

$$P(N(1) = 3) = \frac{e^{-(10)(1)}((10)(1))^3}{3!} = 166.67\, e^{-10}$$

2.

$$P(N(3) > 2) = 1 - P(N(3) \le 2)$$

$$= 1 - [P(N(3) = 0) + P(N(3) = 1) + P(N(3) = 2)]$$

$$= 1 - \left[\frac{e^{-30}(30)^0}{0!} + \frac{e^{-30}(30)^1}{1!} + \frac{e^{-30}(30)^2}{2!}\right]$$

$$= 1 - \left[e^{-30} + 30e^{-30} + 450e^{-30}\right]$$

$$= 1 - 481e^{-30}$$

3.

$$P(N(2) = 3 \mid N(1) = 1) = \frac{P(N(2) = 3, N(1) = 1)}{P(N(1) = 1)}$$

$$= \frac{P(N(2-1) = 3-1)\, P(N(1) = 1)}{P(N(1) = 1)}$$

$$= P(N(1) = 2) = \frac{e^{-(10)(1)}((10)(1))^2}{2!} = 50e^{-10}$$

Problem:

Suppose that minor defects are distributed over the length of a cable as a Poisson process with rate α, and that, independently, major defects are distributed over the cable according to a Poisson process of rate β. Let $N(t)$ be the number of defects, either major or minor, in the cable up to length t. Discuss that $N(t)$ must be a Poisson process of rate $\alpha + \beta$.

Solution

Let $N_1(t)$ and $N_2(t)$ be the minor and major defects independent Poisson processes with rates α and β respectively. Then the number of defects (major or minor) in the cable up to length t is:

$$N(t) = N_1(t) + N_2(t)$$

Since $N_i(t)$, $i = 1, 2$, distributed as Poisson distribution, then the p.g.f. of $N_i(t)$ are:

$$P_{N_1(t)}(S) = \sum_{k=0}^{\infty} p_k S^k = \sum_{k=0}^{\infty} \frac{e^{-\alpha t}(\alpha t)^k}{k!} S^k$$

$$= e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha t S)^k}{k!} = e^{-\alpha t} e^{\alpha t S} = e^{\alpha t(S-1)}$$

And also

$$P_{N_2(t)}(S) = e^{\beta t(S-1)}$$

Then the p.g.f. of $N(t)$ is:

$$P_{N(t)}(S) = P_{N_1(t)}(S)\, P_{N_2(t)}(S)$$

$$= e^{\alpha t(S-1)}\, e^{\beta t(S-1)} = e^{(\alpha+\beta)t(S-1)}$$

Then the distribution of $N(t)$ is a Poisson distribution, then $N(t)$ is a Poisson process with rate $(\alpha + \beta)$.

Problem:

Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes with rates $\lambda_1 = 1$ and $\lambda_2 = 2$ respectively. Let $N(t)$ be the merged process $N(t) = N_1(t) + N_2(t)$.

1. Find the probability that $N(1) = 2$ and $N(2) = 5$.
2. Given that $N(1) = 2$, find the probability that $N_1(1) = 1$.

Solution

$N(t)$ is a Poisson process with rate $\lambda = 1 + 2 = 3$.

1.

$$P(N(1) = 2, N(2) = 5) = P(\text{two arrivals in } (0, 1] \text{ and three arrivals in } (1, 2])$$

$$= \frac{e^{-3\cdot 1}(3 \cdot 1)^2}{2!} \cdot \frac{e^{-3\cdot 1}(3 \cdot 1)^3}{3!} \approx 0.05$$

2.

$$P(N_1(1) = 1 \mid N(1) = 2) = \frac{P(N_1(1) = 1, N(1) = 2)}{P(N(1) = 2)}$$

$$= \frac{P(N_1(1) = 1, N_2(1) = 1)}{P(N(1) = 2)}$$

$$= \frac{P(N_1(1) = 1)\, P(N_2(1) = 1)}{P(N(1) = 2)}$$

$$= \frac{\left[e^{-1} \cdot \frac{1^1}{1!}\right] \left[e^{-2} \cdot \frac{2^1}{1!}\right]}{\frac{e^{-3\cdot 1}(3\cdot 1)^2}{2!}}$$

$$= \frac{4}{9}$$