Lecture 9: Birth and Death Process

Pure Birth Process:

The pure birth process is a type of stochastic process that models the evolution of a population of individuals over time, where each individual can give birth to new individuals at a constant rate. The process is also known as a Yule process or a simple branching process.

In the pure birth process, the birth rate is proportional to the size of the population at each instant in time, which means that the larger the population, the faster it grows. Specifically, if the population size at time t is denoted by N_t, then the birth rate at time t is given by λN_t, where λ is a constant parameter called the birth rate parameter.

Consider the sequence of a positive number $\{\lambda_n\}$. We define a Pure Birth Process as a Linear Markovian Birth Process N_t satisfying the axiom:

1. $P\{N_{t+h} - N_t = N_h = 1\} = \lambda_n h + O(h)$.
2. $P\{N_{t+h} - N_t = N_h = 0\} = 1 - \lambda_n h + O(h)$.
3. $P\{N_{t+h} - N_t = N_h > 1\} = O(h)$.
4. $N_0 = 0$.

Let $P_n(t) = Pr\{N_t = n\}$, then as a Poisson process, we shall have the following equation:

$$P_n(t+h) = P_n(t)(1 - \lambda_n h) + P_{n-1}(t)\lambda_{n-1}h + O(h), \qquad n \geq 1$$

Proceeding as before, we get:

$$P_n'(t) = -\lambda_n P_n(t) + \lambda_{n-1}P_{n-1}(t), \qquad n \geq 1$$

When $(n = 0)$, then:

$$P_0(t+h) = P_0(t)(1 - \lambda_0 h + O(h))$$

$$P_0'(t) = -\lambda_0 P_0(t)$$

For given initial condition, explicit expressions for $P_n(t)$ can be obtained from the above equations.

Yule-Furry Process:

The Yule-Furry process, also known as the Yule process, is a stochastic process that models the evolution of a population of individuals over time, where new individuals can be introduced into the population at a constant rate, in addition to the birth of new individuals.

Pure birth process is called Yule-Furry Process when ($\lambda_n = n\lambda$) and ($\lambda_0 = 0$) which we will be discussed here:

$$P'_n(t) = -n\lambda P_n(t) + (n-1)\lambda P_{n-1}(t), \qquad n \geq 1$$

$$P'_0(t) = 0$$

Starting from a small population size (initial population size) (i), i.e. ($N_0 = i$), then it can be shown that:

$$P_n(t) = \binom{n-1}{n-i}(e^{-\lambda t})^i(1-e^{-\lambda t})^{n-i}$$

Where: $n = i, i+1, i+2, i+3, \ldots$

Which distributed as a Negative Binomial Distribution with parameter ($r = i, p = e^{-\lambda t}$), then the mean and variance of N_t is:

$$E(N_t) = \frac{r}{p} = ie^{\lambda t}$$

$$var(N_t) = \frac{r(1-p)}{p^2} = \frac{i(1-e^{-\lambda t})}{(e^{-\lambda t})^2} = ie^{\lambda t}(e^{\lambda t} - 1)$$

Example:

Suppose that we have a linear Markov birth process (pure birth) in one of the particles with an average birth rate of two births every hour ($\lambda = 2$). Find the probability that the population will be greater than 3 after one hour, and find the expected and variance population size at that time.

Solution:

At ($\lambda = 2$) per hour, and ($t = 1$) with ($i = 1$), then the probability that the population will be greater than 3 after one hour:

$$Pr\{N_t > 3\} = 1 - Pr\{N_t \leq 3\}$$

$$= 1 - \{P_0(1) + P_1(1) + P_2(1) + P_3(1)\}$$

$$P_0(1) = 0$$

$$P_n(t) = \binom{n-1}{n-i}(e^{-\lambda t})^i(1-e^{-\lambda t})^{n-i}$$

$$P_1(1) = \binom{1-1}{1-1}(e^{-2(1)})^1(1-e^{-2(1)})^{1-1} = e^{-2} = 0.135$$

$$P_2(1) = \binom{2-1}{2-1}(e^{-2(1)})^1(1-e^{-2(1)})^{2-1} = e^{-2}(1-e^{-2}) = 0.117$$

$$P_3(1) = \binom{3-1}{3-1}(e^{-2(1)})^1(1-e^{-2(1)})^{3-1} = e^{-2}(1-e^{-2})^2 = 0.101$$

Then:

$$P\{N_t > 3\} = 1 - \{0 + 0.135 + 0.117 + 0.101\} = 0.647$$

The expected population size at time $(t = 1)$ is:

$$E[N_t] = E[N_1] = (1)e^{2(1)} = e^2 = 7.389 \text{ particles.}$$

$$var(N_t) = ie^{\lambda t}(e^{\lambda t} - 1) = (1)e^{2(1)}(e^{2(1)} - 1) = 47.2$$

Pure Death Process:

Consider the sequence of a positive number $\{\mu_n\}$. We define a Pure Death Process as a Linear Markovian Death Process N_t satisfying the axiom:

1. $P\{N_{t+h} - N_t = N_h = 1\} = \mu_n h + O(h)$.
2. $P\{N_{t+h} - N_t = N_h = 0\} = 1 - \mu_n h + O(h)$.
3. $P\{N_{t+h} - N_t = N_h > 1\} = O(h)$.

Let $P_n(t) = Pr\{N_t = n\}$ the probability that the size of the population is n in period $(0, t]$, or (probability of n deaths in the period $(0, t]$), then as a Poisson process, we shall have the following equation:

$$P_i(t+h) = P_i(t)(1 - \mu_i h + O(h)), \qquad i > 1$$

Proceeding as before, we get:

$$P_n(t+h) = P_n(t)(1 - \mu_n h + O(h)) + P_{n+1}(t)(\mu_{n+1} h + O(h)) + \sum_{k=2}^{\infty} P_{n+k}(t)O(h)$$

Dividing the above equations by (h), and taking the limiting when $h \to 0$, we have:

$$P_i'(t) = -\mu_i P_i(t)$$

$$P_n'(t) = \mu_n P_n(t) + \mu_{n+1} P_{n+1}(t)$$

By using the Yule-Furry Process procedure when $(\mu_n = n\mu)$, we have:

$$P_i'(t) = -i\mu P_i(t)$$

$$P_n'(t) = n\mu P_n(t) + (n+1)\mu P_{n+1}(t)$$

By solving the above equations, we have:

$$P_n(t) = \begin{cases} \binom{i}{n}(e^{-\mu t})^n(1-e^{-\mu t})^{i-n}, & n \le i, \\ 0, & n > i. \end{cases}$$

Which distributed as a Binomial Distribution with parameter $(r = i, p = e^{-\mu t})$, then the expected population size is:

$$E(N_t) = r.p = ie^{-\mu t}$$

and the variance population size is:

$$var(N_t) = r.p(1-p) = ie^{-\mu t}(1-e^{-\mu t})$$

Example:

Suppose that the linear Markov death process (pure death) begins in 10 particles with mean weekly death ($\mu = 0.6$). Find the probability that the number of particles will be at least 8 after three days, and find the expected and variance population size at that time.

Solution:

The mean death ($\mu = 0.6$) per week, and ($t = 3/7$) week, with ($i = 10$), then the probability that the number of particles will be at least 8 after three days is:

$$Pr\{N_t \ge 8\} = P_8\left(\frac{3}{7}\right) + P_9\left(\frac{3}{7}\right) + P_{10}\left(\frac{3}{7}\right)$$

$$P_n(t) = \binom{i}{n}(e^{-\mu t})^n(1-e^{-\mu t})^{i-n}$$

$$P_8\left(\frac{3}{7}\right) = \binom{10}{8}\left(e^{-0.6(\frac{3}{7})}\right)^8\left(1-e^{-0.6(\frac{3}{7})}\right)^{10-8} = 0.296$$

$$P_9\left(\frac{3}{7}\right) = \binom{10}{9}\left(e^{-0.6(\frac{3}{7})}\right)^9\left(1-e^{-0.6(\frac{3}{7})}\right)^{10-9} = 0.224$$

$$P_{10}\left(\frac{3}{7}\right) = \binom{10}{10}\left(e^{-0.6(\frac{3}{7})}\right)^{10}\left(1-e^{-0.6(\frac{3}{7})}\right)^{10-10} = 0.076$$

Then:

$$P\{N_t \ge 8\} = 0.296 + 0.224 + 0.076 = 0.596$$

The expected population size at the time $(t = \frac{3}{7})$ is:

$$E[N_t] = E\left[N_{\frac{3}{7}}\right] = (10)e^{-0.6(\frac{3}{7})} = 7.73 \text{ particles.}$$

And the variance in population size at the time $(t = \frac{3}{7})$ is:

$$var(N_t) = ie^{-\mu t}(1 - e^{-\mu t})$$

$$= (10)e^{-0.6(\frac{3}{7})}\left(1 - e^{-0.6(\frac{3}{7})}\right) = 1.75$$

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