

theory

Summation notation and properties

start at this value
go to this value
what to sum

$$\sum_{n=1}^4 n = 1+2+3+4 = 10$$

Σ σ
 S s Sigma is the upper case letter S in Greek. And S stands for Sum.

Example:

The sum $u_1 + u_2 + u_3 + \dots + u_n$ is written in sigma notation as $\sum_{i=1}^n u_i$.

Example: Evaluate $\sum_{r=1}^4 r^3$

Solution:

$$\begin{aligned}\sum_{r=1}^4 r^3 &= 1^3 + 2^3 + 3^3 + 4^3 \\ &= 1 + 8 + 27 + 64 = 100\end{aligned}$$

Example: Evaluate $\sum_{n=2}^5 n^2$

Solution:

$$\begin{aligned}\sum_{n=2}^5 n^2 &= 2^2 + 3^2 + 4^2 + 5^2 \\ &= 4 + 9 + 16 + 25 = 54\end{aligned}$$

theory

Example: Evaluate $\sum_{k=0}^5 2^k$

Solution:

$$\begin{aligned}\sum_{k=0}^5 2^k &= 2^0 + 2^1 + 2^2 + 2^3 + 2^4 + 2^5 \\ &= 1 + 2 + 4 + 8 + 16 + 32 \\ &= 63\end{aligned}$$

Example: Evaluate $\sum_{r=1}^4 (-1)^r$

Solution: here, we need to remember that $(-1)^2 = +1$, $(-1)^3 = -1$, and so on,
so

$$\begin{aligned}\sum_{r=1}^4 (-1)^r &= (-1)^1 + (-1)^2 + (-1)^3 + (-1)^4 \\ &= (-1) + 1 + (-1) + 1 = 0\end{aligned}$$

Example: Evaluate $\sum_{k=1}^3 \left(-\frac{1}{k}\right)^2$

Solution: once again, we must remebmer how to deal with powers of -1:

$$\begin{aligned}\sum_{k=1}^3 \left(-\frac{1}{k}\right)^2 &= \left(-\frac{1}{1}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{3}\right)^2 \\ &= 1 + \frac{1}{4} + \frac{1}{9} =\end{aligned}$$

theory

Example: write the sum

$$-1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots + \frac{1}{100}$$

In sigma notation.

Solution: we can now see that k -th term is $(-1)^k \frac{1}{k}$, and that there are 100 term,
so we would write the sum in sigma notation as

$$\sum_{k=1}^{100} (-1)^k \frac{1}{k}$$

Rules for use with sigma notation

If a and c are constants, and if $f(k)$ and $g(k)$ are functions of k , then

$$(1) \quad \sum_{k=1}^n c = c + c + c + \cdots + c = nc$$

$$(2) \quad \sum_{k=1}^n ck = (c \times 1) + (c \times 2) + \cdots (c \times n) \\ = c \times (1 + \cdots + n)$$

$$c \sum_{k=1}^n k$$

$$(3) \quad \sum_{k=1}^n (k + c) = (1 + c) + (2 + c) + \cdots + (n + c) \\ (c + c + \cdots + c) + (1 + 2 + \cdots + n) \\ = nc + \sum_{k=1}^n k$$

theory

$$(4) \quad \sum_{k=1}^n (ag(k) + c) = nc + a \sum_{k=1}^n g(k)$$

$$(5) \quad \sum_{k=1}^n (f(k) + g(k)) = \sum_{k=1}^n f(k) + \sum_{k=1}^n g(k)$$

Extra Examples:

$$(1) \quad \sum_{n=1}^4 (2n + 1) = 3 + 5 + 7 + 9 = 24$$

$$(2) \quad \sum_{i=1}^3 i(i + 1) = 1 \times 2 + 2 \times 3 + 3 \times 4 = 20$$

$$(3) \quad \sum_{i=3}^5 \frac{i}{i+1} = \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$$

$$(4) \quad (x_1 - \mu)^2 + (x_2 - \mu)^2 + (x_3 - \mu)^2 + (x_4 - \mu)^2 = \sum_{k=1}^4 (x_k - \mu)^2$$

Note:

If $f(i)$ represents some expression (function) involving i , then

$$\sum_{i=1}^n f(i) = f(1) + f(2) + f(3) + \cdots + f(n)$$

Example:

$$\begin{aligned} \sum_{i=1}^4 (2 + i^2) &= (2 + 1^2) + (2 + 2^2) + (2 + 3^2) + (2 + 4^2) \\ &= 3 + 6 + 11 + 18 = 38 \end{aligned}$$

theory

The well-known summation rules:

1. $\sum_{i=1}^n c = c + c = c = \dots + c$ (n times)=nc, where c is a constant
2. $\sum_{i=1}^n i = 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$
3. $\sum_{i=1}^n i^2 = 1^2 + 2^2 + 3^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$
4. $\sum_{i=1}^n i^3 = 1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{n^2(n+1)^2}{4}$

Arithmetic and geometric series

two common types of mathematical sequences are arithmetic sequences and geometric sequences. An arithmetic sequence has a constant difference between each consecutive pair of terms. This is similar to the linear functions that have the form $y = mx + b$. A geometric sequence has a constant ratio between each pair of consecutive terms. This would create the effect of a constant multiplier.

Examples

Arithmetic Sequence:

{5,11,17,23,29,35,...}

Notice here the constant difference is 6. If we wanted to write a general term for this sequence, there are several approaches. One approach is to take the constant difference as the coefficient for the n term: $a_n = 6n + ?$ Then we just need to fill in the question mark with a value that matches the sequence. We could say for the sequence:

{5,11,17,23,29,35,...}

theory

$$a_n = 6n - 1$$

There is also a formula which you can memorize that says that any arithmetic sequence with a constant difference d is expressed as:

$$a_n = a_1 + (n - 1)d$$

Notice that if we plug in the values from our example, we get the same answer as before:

$$a_n = a_1 + (n - 1)d$$

$$a_1 = 5, d = 6$$

$$\text{So, } a_1 + (n - 1)d = 5 + (n - 1) \times 6 = 5 + 6n - 6 = 6n - 1$$

If the terms of an arithmetic sequence are getting smaller, then the constant difference is a negative number.

$$\{24, 19, 14, 9, 4, -1, -6, \dots\}$$

$$a_n = -5n + 29$$

Geometric Sequence

In a geometric sequence there is always a constant multiplier. If the multiplier is greater than 1, then the terms will get larger. If the multiplier is less than 1, then the terms will get smaller.

$$\{2, 6, 18, 54, 162, \dots\}$$

theory

Notice in this sequence that there is a constant multiplier of 3. This means that 3 should be raised to the power of n in the general expression for the sequence. The fact that these are not multiples of 3 means that we must have a coefficient before the 3^n

$$\{2, 6, 18, 54, 162, \dots\}$$

$$a_n = 2 * 3^{n-1}$$

If the terms are getting smaller, then the multiplier would be in the denominator:

$$\{50, 10, 2, 0.4, 0.08, \dots\}$$

Notice here that each term is begin divided by 5 (or multiplied by $\frac{1}{5}$).

$$\{50, 10, 2, 0.4, 0.08, \dots\}$$

$$a_n = \frac{50}{5^{n-1}} \text{ or } a_n = \frac{250}{5^n} \text{ or } a_n = 50 * \left(\frac{1}{5}\right)^{n-1}$$

Exercises :

Determine whether each sequence is arithmetic, geometric or neither. If it is arithmetic, determine the constant difference. If it is geometric determine the constant ratio.

1) $\{18, 22, 26, 30, 34, \dots\}$

2) $\{9, 19, 199, 1999, \dots\}$

3) $\{8, 12, 18, 27, \dots\}$

theory

4) $\{15, 7, -1, -9, -17, \dots\}$

5) $\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \dots\}$

6) $\{100, -50, 25, -12.5, \dots\}$