قسم الرياضيات المرحلة الثانية

Advanced Calculus

التفاضل المتقدم

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Lecture 1

Sequence

Part 1

$\{a_n\}_1^\infty$

$\langle a_1, a_2, a_3, \ldots, a_n, \ldots \rangle$

What is a sequence?

- Consider a set $A = \{a\ \ ,a\ \ ,a\ \ ,a\ \ ,\dots,a\ \ ,\dots\}$
- Adding order $a\ \ ,\quad a\ \ ,\quad a\ \ ,\quad a\ \ ,\dots,\quad a\ \ \dots,$

  $1 \qquad 2 \qquad 3 \qquad 4 \qquad n$

- The order $\langle a_1, a_2, a_3, \dots, a_n, \dots \rangle$ is called a **sequence**

What is a sequence?

- **Definition**: A ***Sequence*** is a function whose **domain** is the set of **positive integers** $\mathbb{Z}_+ = \{1,2,3,4,\cdots\}$.

- We denote to sequences as $\{\boldsymbol{a_n}\}_{\mathbf{1}}^{\infty}$ or $\langle \boldsymbol{a_n} \rangle$

$$\{a_n\}_1^{\infty} = \langle a_n \rangle = \langle a_1, a_2, a_3, \dots, a_n, \dots \rangle$$

- **Example**: $\{n^2\}_1^{\infty} = \langle 1, 4, 9, 16, \dots, n^2, \dots \rangle$

Monotonicity

$$a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots \quad \forall n \geq 1 \Rightarrow \text{ increasing seq.}$$

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots \quad \forall n \geq 1 \Rightarrow \text{ nondecreasing seq.}$$

$$a_1 > a_2 > \cdots > a_n > a_{n+1} > \cdots \quad \forall n \geq 1 \Rightarrow \text{ decreasing seq.}$$

$$a_1 \geq a_2 \geq \cdots \geq a_n \geq a_{n+1} \geq \cdots \quad \forall n \geq 1 \Rightarrow \text{ nonincreasing seq.}$$

increasing ⇒ nondecreasing ⇒ **monotonic** ⇐ nondecreasing ⇐ decreasing

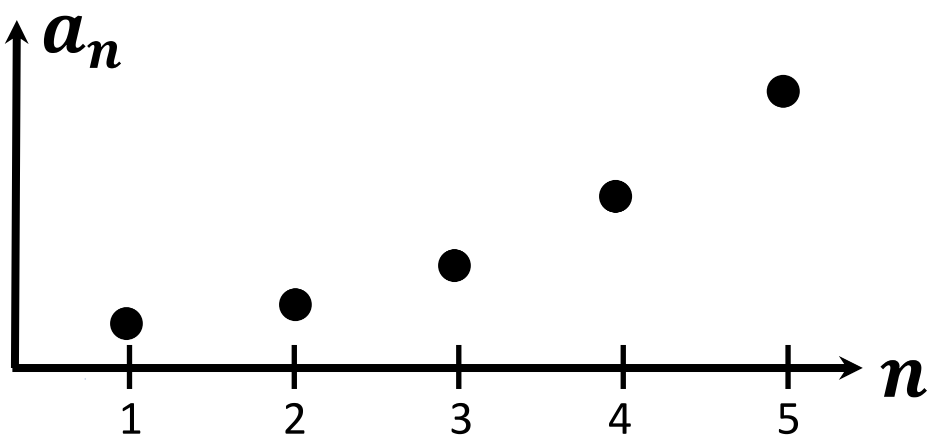
Not monotonic

Examples

- $\{n^2\}_1^\infty = \langle 1, 4, 9, \dots, n^2, \dots \rangle$

$$1 < 4 < 9 < 16 < 25 < \cdots$$

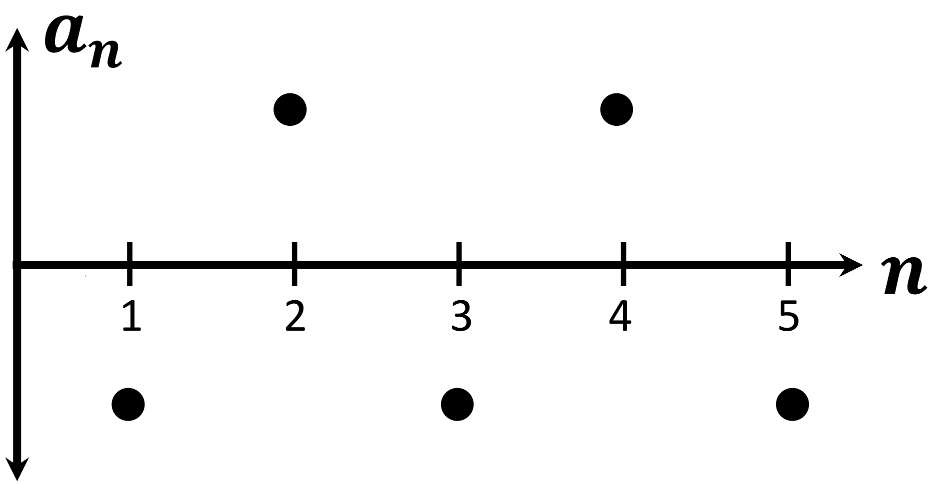
$$n^2 < (n+1)^2 \quad \forall n \geq 1$$



Increasing $\Rightarrow$ Monotonoic

- $\langle (-1)^n \rangle = \langle -1, +1, -1, \dots \rangle$

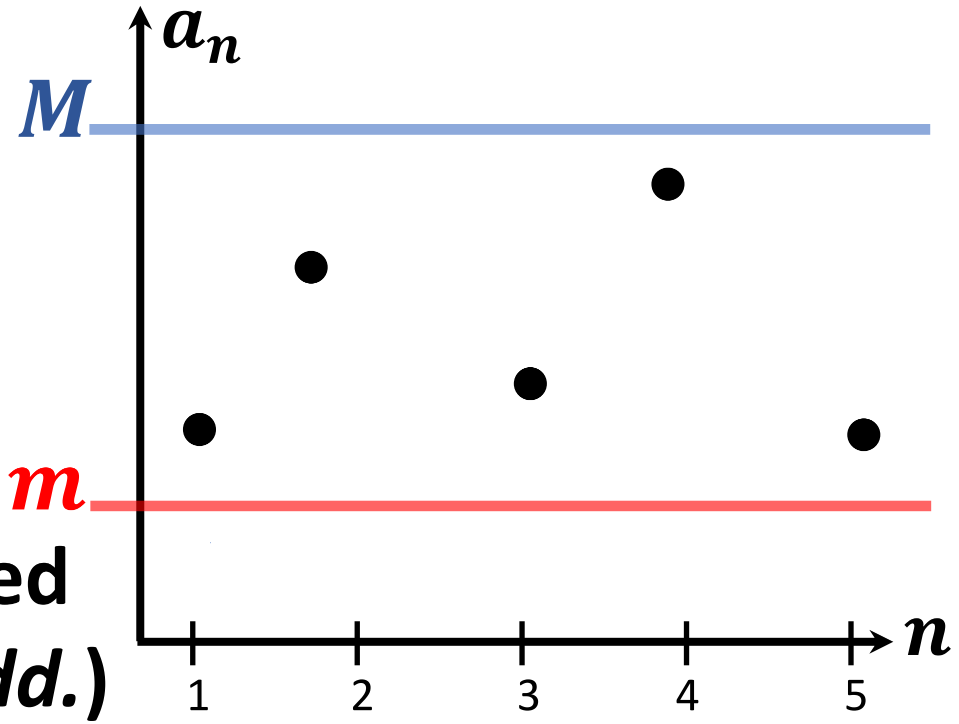
$$-1 < +1 > -1 < +1 > \cdots$$



not Monotonoic

Bounded sequence

- If $\exists \boldsymbol{M} \in \mathbb{R}$ such that $\boldsymbol{a_n} \leq \boldsymbol{M} \;\; \forall n \geq 1$ then $\langle a_n \rangle$ is called ***bounded above***
- If $\exists \boldsymbol{m} \in \mathbb{R}$ such that $\boldsymbol{a_n} \geq \boldsymbol{m} \;\; \forall n \geq 1$ then $\langle a_n \rangle$ is called ***bounded below***
- If $\langle a_n \rangle$ is **bounded below AND bounded above** then $\langle a_n \rangle$ is called ***bounded* (*bdd.*)**
- If $\langle a_n \rangle$ is **not bounded below OR not bounded above** then $\langle a_n \rangle$ is called ***not bounded* (not *bdd.*)**



Examples

- $\{n^2\}_1^\infty = \langle 1, 4, 9, \dots, n^2, \dots \rangle$

$$\left.\begin{array}{l} \because\ n^2 \geq 1 \quad \forall n \geq 1 \implies \textbf{bdd. below by } \boldsymbol{m} = \mathbf{1} \\ \because\ n^2 \to \infty \text{ as } n \to \infty \implies \textbf{not bdd. above} \end{array}\right\} \implies \begin{array}{c} \{n^2\}_1^\infty \\ \textbf{not bdd.} \end{array}$$

- $\{1/n\}_1^\infty = \langle 1, 1/2, 1/3, \dots, 1/n, \dots \rangle$

$$\left.\begin{array}{l} \because\ 1/n \leq 1 \quad \forall n \geq 1 \implies \textbf{bdd. above by } \boldsymbol{M} = \mathbf{1} \\ \because\ 1/n \geq 0 \quad \forall n \geq 1 \implies \textbf{bdd. below by } \boldsymbol{m} = \mathbf{0} \end{array}\right\} \implies \begin{array}{c} \{1/n\}_1^\infty \\ \textbf{bdd.} \end{array}$$

Convergence

- A sequence $\langle a_n \rangle$ is called ***convergent* (*conv.*)** if $\exists \boldsymbol{L} \in \mathbb{R}$ s.t.

$$\boldsymbol{\lim_{n\to\infty} a_n = L}$$

- ***Definition:*** $\lim_{n\to\infty} a_n = L$ if and only if $\forall \boldsymbol{\epsilon} > \boldsymbol{0}\ \exists \boldsymbol{N} \in \mathbb{Z}_+$ *s.t.*

$$\boldsymbol{|a_n - L| < \epsilon \quad \forall n > N}$$

- A sequence $\langle a_n \rangle$ is called ***divergent* (*div.*)** if it has **no limit**

$$\lim_{n\to\infty} a_n = \pm\infty \quad \textbf{OR} \quad \lim_{n\to\infty} a_n \text{ is } \textbf{not unique}$$

Examples

- $\{1/n\}_1^\infty = \langle 1, 1/2\,, 1/3, \dots, 1/n\,, \dots \rangle$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1/n = 1/\infty = 0 \qquad \Longrightarrow \textbf{ has a limit } \boldsymbol{L = 0}$$

$$\therefore\ \{1/n\}_1^\infty \text{ is } \textbf{conv.}$$

- $\langle (-1)^n \rangle = \langle -1, +1, -1, \dots \rangle$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (-1)^n = \left\{\begin{matrix} +1 & n \text{ is even} \\ -1 & n \text{ is odd} \end{matrix}\right\} \Longrightarrow \textbf{ has no limit}$$

$$\therefore\ \{(-1)^n\}_1^\infty \text{ is } \textbf{div.}$$

Exercises

➢ Show whether the following sequences are monotonic, bounded, convergent or not:

$$\{-1/n\}_1^\infty, \qquad \{1+(-1)^n\}_1^\infty, \qquad \{(-1)^n/n\}_1^\infty$$

$$\left\langle\left[\frac{n+1}{2}\right]\right\rangle, \qquad \left\langle\frac{n}{n+1}\right\rangle, \qquad \left\langle\sin\left(\frac{1}{n}\right)\right\rangle$$

➢ Give examples of sequences satisfying the following:

- Bounded and not monotonic.
- Increasing and convergent.
- Monotonic and divergent.

ترجمة مصطلحات

- Sequence متتابعة
- Function دالة
- Domain منطلق
- Positive integers الأعداد الصحيحة الموجبة
- Denote نرمز
- Monotonic رتيبة
- Increasing متزايدة
- Nonincreasing غير متزايدة
- Decreasing متناقصة
- Nondecreasing غير متناقصة
- Bounded مقيدة
- Bounded above مقيدة من الأعلى
- Bounded below نقيدة من الأدنى
- Convergent متقاربة
- Divergent متباعدة
- Even زوجي
- Odd فردي

Lecture 2

Sequence

Part 2

$$\lim_{n \to \infty} a_n = L$$

Determinate forms

- Let $r > 0$, the following forms are determinate:
$$\frac{r}{0} = \infty, \infty \mp r = \infty, \infty \cdot r = \infty, \frac{\infty}{r} = \infty, \frac{r}{\infty} = 0, \infty^r = \infty$$
- the following forms are also determinate:
$$\infty + \infty = \infty, \infty \cdot \infty = \infty, \infty^{\infty} = \infty, \frac{\infty}{0} = \infty, \frac{0}{\infty} = 0$$
- Let $r \geq 0$, then
$$\lim_{n \to \infty} r^n = r^{\infty} = \begin{Bmatrix} \infty & r > 1 \\ 1 & r = 1 \\ 0 & 0 \leq r < 1 \end{Bmatrix}$$

Indeterminate forms

Indeterminate forms: $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty, \infty - \infty, 0^0, \infty^0, 1^\infty$

$$\frac{0}{0} \stackrel{?}{=} \begin{cases} 0 \\ 1 \\ \infty \end{cases} \qquad \frac{\infty}{\infty} \stackrel{?}{=} \begin{cases} \infty \\ 1 \\ 0 \end{cases} \qquad 0 \cdot \infty \stackrel{?}{=} \begin{cases} 0 \\ \infty \end{cases}$$

$$\infty^0 \stackrel{?}{=} \begin{cases} \infty \\ 1 \end{cases} \qquad 0^0 \stackrel{?}{=} \begin{cases} 0 \\ 1 \end{cases} \qquad \infty - \infty \stackrel{?}{=} \begin{cases} \infty \\ 0 \\ -\infty \end{cases}$$

(1) $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty$

Example: $\left\langle \frac{n^2+2n-1}{3n^3+n} \right\rangle \Rightarrow \lim_{n\to\infty} \frac{n^2+2n-1}{3n^3+n} = \frac{\infty^2+2\infty-1}{3\infty^3+\infty} = \frac{\infty}{\infty}$?

By cancelling the greatest power n^3

$$\lim_{n\to\infty} \frac{n^2+2n-1}{3n^3+n} \overset{\div n^3}{=} \lim_{n\to\infty} \frac{\frac{n^2}{n^3}+2\frac{n}{n^3}-\frac{1}{n^3}}{3\frac{n^3}{n^3}+\frac{n}{n^3}} = \lim_{n\to\infty} \frac{\frac{1}{n}+2\frac{1}{n^2}-\frac{1}{n^3}}{3+\frac{1}{n^2}}$$

$$= \frac{\frac{1}{\infty}+2\frac{1}{\infty^2}-\frac{1}{\infty^3}}{3+\frac{1}{\infty^2}} = \frac{0+0-0}{3+0} = 0 \quad \Rightarrow \text{Conv.}$$

(1) $\frac{0}{0}, \frac{\infty}{\infty}, 0 \cdot \infty$

Example: $\langle n \sin(\frac{1}{n}) \rangle \Rightarrow \lim_{n\to\infty} n \sin(\frac{1}{n}) = \infty \sin(\frac{1}{\infty}) = \infty \sin(0) = \infty \cdot 0$?

L'Hôpital's Rule: $\lim_{n\to\infty} \frac{f(n)}{g(n)} = \frac{0}{0}$ or $\frac{\infty}{\infty} \quad \Longrightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$

$$\lim_{n\to\infty} n \sin\left(\frac{1}{n}\right) = \lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \frac{\sin\left(\frac{1}{\infty}\right)}{\frac{1}{\infty}} = \frac{\sin(0)}{0} = \frac{0}{0}$$?

L'Hôpital's Rule

d/dn

$$\lim_{n\to\infty} \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{\cos\left(\frac{1}{n}\right)\cdot\left(\frac{-1}{n^2}\right)}{\left(\frac{-1}{n^2}\right)} = \lim_{n\to\infty} \cos\left(\frac{1}{n}\right) = \cos\left(\frac{1}{\infty}\right) = \cos(0) = 1$$ ✓

$\Longrightarrow \langle n \sin(\frac{1}{n}) \rangle$ Conv.

(2) $\infty - \infty$

Example: $\langle n^3 - 2n \rangle \Rightarrow \lim_{n\to\infty} n^3 - 2n = \infty^3 - 2\infty = \infty - \infty$?

Factorizing: factoring out the greatest common

$$\lim_{n\to\infty} n^3 - 2n \overset{\text{Factorizing}}{=} \lim_{n\to\infty} n(n^2 - 2) = \infty\,(\infty^2 - 2)$$

$$= \infty \cdot \infty = \infty \checkmark$$

$$\Rightarrow \langle n^3 - 2n \rangle \text{ div.}$$

(2) $\infty - \infty$

Example: $\langle \sqrt{n+1} - \sqrt{n} \rangle \Rightarrow \lim\limits_{n\to\infty} \sqrt{n+1} - \sqrt{n} = \sqrt{\infty+1} - \sqrt{\infty} = \infty - \infty$?

conjugate: $(a+b) \xleftrightarrow{\text{cojugate}} (a-b) \Longrightarrow (a+b)\cdot(a-b) = a^2 - b^2$

×÷ conjugate

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} \times \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{\left(\sqrt{n+1}\right)^2 - (\sqrt{n})^2}{\sqrt{n+1}+\sqrt{n}}$$

$$= \lim_{n\to\infty} \frac{(n+1)-n}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{1}{\sqrt{\infty+1}+\sqrt{\infty}}$$

$$= \frac{1}{\infty+\infty} = \frac{1}{\infty} = 0 \checkmark \Longrightarrow \langle \sqrt{n+1} - \sqrt{n} \rangle \text{ Conv.}$$

(3) $0^0, \infty^0, 1^\infty$

Example: $\left\langle \left(\frac{n+1}{n}\right)^n \right\rangle \Rightarrow \lim\limits_{n\to\infty} \left(\frac{n+1}{n}\right)^n = \lim\limits_{n\to\infty} \left(1+\frac{1}{n}\right)^n = \left(1+\frac{1}{\infty}\right)^\infty = 1^\infty$?

Logarithm: $\ln(a^b) = b\ln(a)$, $e^{\ln(a)} = a$

Let $a_n = \left(1+\frac{1}{n}\right) \overset{ln}{\Longrightarrow} \ln(a_n) = \ln\left(1+\frac{1}{n}\right)^n = n\ln\left(1+\frac{1}{n}\right)$

$$\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} n\ln\left(1+\frac{1}{n}\right) = \infty \ln\left(1+\frac{1}{\infty}\right) = \infty \overset{?}{\cdot} 0$$

L'Hôpital's Rule

$$\lim_{n\to\infty} \ln(a_n) = \lim_{n\to\infty} \frac{\ln\left(1+\frac{1}{n}\right)}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1/\left(1+\frac{1}{n}\right)\cdot\frac{-1}{n^2}}{\frac{-1}{n^2}} = 1/\left(1+\frac{1}{\infty}\right) = 1$$

$$\Longrightarrow \lim_{n\to\infty}(a_n) = e^{\lim\limits_{n\to\infty}\ln(a_n)} = e^1 = e \checkmark$$

$$\Longrightarrow \left\langle \left(\frac{n+1}{n}\right)^n \right\rangle \to e \ \text{Conv.}$$

Exercises

- Show whether the following sequences are convergent or divergent:

$$\left\{\left(-\frac{1}{2}\right)^n\right\}_{n=1}^{\infty} \qquad \left\{\frac{2n^4-n^3+n}{n^4+n-1}\right\}_{n=1}^{\infty} \qquad \left\langle\frac{\sin(n)}{n}\right\rangle \qquad \left\langle\frac{\ln(n)}{n^2}\right\rangle$$

$$\langle e^n-2^n\rangle \qquad \left\{\left(\frac{n}{n+1}\right)^n\right\}_{n=1}^{\infty} \qquad \left\langle\frac{n^2}{e^n}\right\rangle \qquad \langle\sqrt[n]{n}\,\rangle$$

ترجمة مصطلحات

Determinate forms	صيغ معيّنة (محددة الاجابة)
Indeterminate forms	صيغ غير معيّنة (غير محددة الاجابة)
Cancelling	حذف، اختزال
Greatest power	أعلى درجة، أعظم أس
L'Hôpital's Rule	قاعدة لوبتال
Factorizing	تحليل الى عوامل
greatest common factor	العامل المشترك الأكبر
Conjugate	مُرافق الحدانية
Logarithm	لوغاريتم

Lecture 3

Infinite Series المتسلسلات غير المنتهية

Definition: The infinte series is an expression of the form $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$

a_n : The nth term الحد النوني

$S_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_n}_{n \text{ terms}}$ nth partial sum المجموع الجزئي النوني

$\langle S_n \rangle = \langle S_1, S_2, S_3, \cdots, S_n, \cdots \rangle$ the sequence of of partial sums متتابعة المجاميع الجزئية

Definition:

The infinite series $\sum_{n=1}^{\infty} a_n$ is called convergent and has the sum S if the seq. of partial sums $\langle S_n \rangle$ is convergent to the limit S i.e.

$$\lim_{n\to\infty} S_n = S \implies \sum_{n=1}^{\infty} a_n = S \text{ conv.}$$

if the series $\sum_{n=1}^{\infty} a_n$ is not convergent, then $\sum_{n=1}^{\infty} a_n$ is called divergent.

divergent series has no sum.

Geometric Series المتسلسلة الهندسية

Def:- The Series $\sum a_n$ is called geometric series if it can be written in the form:

$$\sum_{n=1}^{\infty} a r^{n-1} = a + ar + ar^2 + \dots + ar^{n-1} + \dots$$

where $a = a_1$ is the coefficient المعامل

ثوابت
constants $\leftarrow r = \frac{a_2}{a_1}$ is the ratio النسبة

Theorem: Consider the geometric series $\sum_{n=1}^{\infty} a r^{n-1}$

① if $\boxed{|r| < 1} \Rightarrow \boxed{\text{Conv.}}$

② if $\boxed{|r| \geq 1} \Rightarrow \boxed{\text{div.}}$

the convergent geometric series has the sum

Sum $\boxed{S = \frac{a}{1-r}}$

Proof:

case 1: if $|r| = 1 \Rightarrow r = 1$ or $r = -1$

① if $\boxed{r = 1}$

$$\therefore \sum_{n=1}^{\infty} a(1)^{n-1} = a + a + a + \dots + a + \dots = \sum_{n=1}^{\infty} a$$

$$S_n = a_1 + a_2 + \dots + a_n$$
$$= \underbrace{a + a + \dots + a}_{n \text{ times}} = na$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} na = \infty \cdot a = \mp\infty \Rightarrow \langle S_n \rangle \text{ div.}$$

$$\Rightarrow \sum_{n=1}^{\infty} a \quad \boxed{\text{div.}}$$

② if $\boxed{r=-1}$

$$\sum_{n=1}^{\infty} a(-1)^{n-1} = a - a + a - a + \cdots + (-1)^{n-1} a + \cdots$$

$$S_1 = a$$

$$S_2 = a - a = 0$$

$$S_3 = a - a + a = a$$

$$S_4 = a - a + a - a = 0$$

$$\therefore S_n = \begin{cases} a & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$\therefore \lim_{n\to\infty} S_n = \begin{cases} a & n \text{ odd} \\ 0 & n \text{ even} \end{cases} \xRightarrow{\text{غير موجودة}} \langle S_n \rangle \text{ div.}$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n a \quad \boxed{\text{div.}}$$

Case 2: if $|r| \neq 1 \Rightarrow |r| > 1$ or $|r| < 1$

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}}$$

$$\mp r S_n = \mp \cancel{ar} \mp \cancel{ar^2} \mp \cancel{ar^3} + \cdots \mp ar^n$$

$$S_n - rS_n = a - ar^n$$

$$S_n(1-r) = a(1-r^n) \qquad \div (1-r)$$

$$S_n = \frac{a}{1-r}(1-r^n)$$

① if $\boxed{|r|>1} \Rightarrow r^{\infty} = \mp\infty$

$$\therefore \lim_{n\to\infty} S_n = \frac{a}{1-r}(1-r^{\infty}) = \pm\infty \Rightarrow \text{div.}$$

② if $\boxed{|r|<1} \Rightarrow r^{\infty} = 0$

$$\therefore \lim_{n\to\infty} S_n = \frac{a}{1-r}(1-r^{\infty}) = \frac{a}{1-r}(1-0) = \boxed{\frac{a}{1-r} = S} \quad \boxed{\text{conv.}}$$

Example: $\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n}$

Sol:-
$$\sum_{n=1}^{\infty} \frac{(-1)^n}{2^n} = \frac{-1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n}{2^n} + \cdots$$

$$a = a_1 = \boxed{-\frac{1}{2} = a}, \quad r = \frac{a_2}{a_1} = \frac{\frac{1}{4}}{-\frac{1}{2}} = \frac{1}{4} * \frac{2}{-1} = \boxed{\frac{-1}{2} = r}$$

$$\because |r| = \left|-\frac{1}{2}\right| = \frac{1}{2} < 1 \Rightarrow \boxed{\text{Conv.}}$$

$$\boxed{S = \frac{a}{1-r}} \Rightarrow S = \frac{-\frac{1}{2}}{1-(\frac{-1}{2})} = \frac{-\frac{1}{2}}{\frac{3}{2}} = \boxed{\frac{-1}{3} = S}$$

Theorem:- If $\sum a_n$ is $\boxed{\text{conv.}} \Longrightarrow \boxed{\lim_{n\to\infty} a_n = 0}$

Proof:- Let $\sum a_n$ be conv. $\Rightarrow \langle S_n \rangle$ is conv.

$$\therefore \lim_{n\to\infty} S_n = S$$

$$S_n = \underbrace{a_1 + a_2 + a_3 + \cdots + a_{n-1}}_{S_{n-1}} + a_n$$

$$S_n = S_{n-1} + a_n \Longrightarrow a_n = S_n - S_{n-1}$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - S_{n-1} = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$
$$= S - S = 0$$

$$\therefore \sum a_n \text{ conv.} \underset{\not\longleftarrow}{\longrightarrow} \lim_{n\to\infty} a_n = 0$$

$\boxed{T_1}$ Divergence Test   اختبار التباعد

بأخذ المعاكس الإيجابي للبرهنة اعلاه نستنتج

① if $\boxed{\lim_{n\to\infty} a_n \neq 0} \Longrightarrow \sum a_n$ $\boxed{\text{div.}}$

② if $\lim_{n\to\infty} a_n = 0 \Longrightarrow$ fails   يفشل الاختبار

Example: $\sum_{n=0}^{\infty} \frac{n}{n+1}$

Sol: $a_n = \frac{n}{n+1}$ By Div. Test

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n+1} \overset{\frac{d}{dn}}{=} \lim_{n\to\infty} \frac{1}{1} = 1 \quad \boxed{\text{L'Hopital}}$$

$$\because \lim_{n\to\infty} a_n = 1 \neq 0 \Rightarrow \sum_{n=0}^{\infty} \frac{n}{n+1} \ \boxed{\text{div.}}$$

$\boxed{T_2}$ Integral Test اختبار التكامل

Let $\sum_{n=1}^{\infty} a_n$ be an infinite series with positive terms $a(n) = a_n > 0 \ \forall n \geq 1$, then

① if $\lim_{b\to\infty} \int_1^b a(x)\,dx = I < \infty \Rightarrow \sum a_n \ \boxed{\text{conv.}}$

② if $\lim_{b\to\infty} \int_1^b a(x)\,dx = I = \infty \Rightarrow \sum a_n \ \boxed{\text{div.}}$

Example: $\sum_{n=0}^{\infty} \frac{n}{e^{n^2}} \Rightarrow a_n = \frac{n}{e^{n^2}} = n e^{-n^2}$

Sol: Using the Integral Test

$$I = \lim_{b\to\infty} \int_0^b x e^{-x^2}\,dx = \lim_{b\to\infty} \int_0^b -\frac{1}{2}(-2x) e^{-x^2}\,dx$$

$$= \lim_{b\to\infty} -\frac{1}{2}\left[e^{-x^2}\right]_0^b = \lim_{b\to\infty} -\frac{1}{2}\left[e^{-b^2} - e^{-0^2}\right]$$

$$= -\frac{1}{2}\left[e^{-\infty^2} - e^{0^2}\right] = -\frac{1}{2}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{2}\left[\frac{1}{e^{\infty}} - 1\right]$$

$$\therefore I = -\frac{1}{2}\left[\frac{1}{\infty} - 1\right] = -\frac{1}{2}[0-1] = \boxed{+\frac{1}{2} < \infty} \Rightarrow \sum \frac{n}{e^{n^2}} \ \boxed{\text{Conv.}}$$

P-Series

المتسلسلة من النظام P

Def: The series $\sum a_n$ is called P-series if it can be written in the form

$$\sum_{n=1}^{\infty} \frac{1}{n^P} = 1 + \frac{1}{2^P} + \frac{1}{3^P} + \frac{1}{4^P} + \cdots + \frac{1}{n^P} + \cdots$$

where $P > 0$ is a positive constant.

Theorem: Consider the P-series $\sum_{n=1}^{\infty} \frac{1}{n^P}$

① if $\boxed{P \leq 1} \Rightarrow \boxed{\text{div.}}$

② if $\boxed{P > 1} \Rightarrow \boxed{\text{conv.}}$

Proof: Using the Integral Test

$$a_n = \frac{1}{n^P} \Rightarrow a(x) = \frac{1}{x^P} = x^{-P}$$

Case 1: if $\boxed{P = 1}$

متسلسلة توافقية

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots \quad \boxed{\text{Harmonic Ser.}}$$

$$I = \lim_{b \to \infty} \int_1^b \frac{1}{x}\, dx = \lim_{b \to \infty} \left[\ln x\right]_1^b$$

$$= \lim_{b \to \infty} \ln(b) - \ln(1)$$

$$= \ln(\infty) - \ln(1)$$

$$= \infty - 0 = \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n} \text{ is } \boxed{\text{div.}}$$

case 2: if $\boxed{p \neq 1}$

$$I = \lim_{b\to\infty} \int_1^b a(x)\,dx = \lim_{b\to\infty} \int_1^b x^{-p}\,dx$$

$$= \lim_{b\to\infty} \left[\frac{x^{-p+1}}{-p+1}\right]_1^b$$

$$= \lim_{b\to\infty} \frac{1}{1-p}\left[b^{1-p} - 1^{1-p}\right]$$

① if $\boxed{p<1}$ $\Rightarrow 1-p>0$

$$\therefore I = \lim_{b\to\infty} \frac{1}{1-p}\left[b^{1-p}-1\right] = \frac{1}{1-p}\left[\infty^{1-p}-1\right] = \frac{1}{1-p}\left[\infty - 1\right]$$

$$\because I = \lim_{b\to\infty}\int_1^b a(x)\,dx = \infty \Longrightarrow \sum_{n=1}^{\infty}\frac{1}{n^p}\ \boxed{\text{div.}}$$

② if $\boxed{p>1}$ $\Rightarrow 1-p<0 \Rightarrow p-1>0$

$$\therefore I = \lim_{b\to\infty}\frac{1}{1-p}\left[b^{1-p}-1\right] = \frac{1}{1-p}\left[\infty^{1-p}-1\right] = \frac{-1}{p-1}\left[\frac{1}{\infty^{p-1}}-1\right]$$

$$= \frac{-1}{p-1}\left[\frac{1}{\infty}-1\right] = \frac{-1}{p-1}\left[0-1\right] = \frac{1}{p-1}$$

$$\because I = \lim_{b\to\infty}\int_1^b a(x)\,dx = \boxed{\frac{1}{p-1}<\infty} \Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^p}\ \boxed{\text{Conv.}}$$

Example: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

Sol: $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty}\frac{1}{n^{\frac{1}{2}}}$ $\Rightarrow$ p-series $\sum_{n=1}^{\infty}\frac{1}{n^p}$

$$p = \frac{1}{2} < 1 \Rightarrow \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\ \boxed{\text{div.}}$$

Exercise:

Show whether the following series are convergent or divergent:

$\sum_{n=0}^{\infty} n^2$	$\sum_{n=1}^{\infty} n^{-3}$	$\sum_{n=0}^{\infty} 3^{-n}$	$\sum_{n=1}^{\infty} \frac{n}{(n^2+1)^2}$
$\sum_{n=1}^{\infty} \sqrt[3]{\frac{1}{n^2}}$	$\sum_{n=0}^{\infty} \frac{e^n}{2^n}$	$\sum_{n=0}^{\infty} \sin(n)$	$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

Lecture 4

T3 Root Test

اختبار الجذر

Let $\sum_{n=1}^{\infty} a_n$ be a series and let

$$\boxed{\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}}$$

① if $\boxed{\rho < 1} \Rightarrow \sum_{n=1}^{\infty} a_n$ $\boxed{\text{Conv.}}$

② if $\boxed{\rho > 1} \Rightarrow \sum_{n=1}^{\infty} a_n$ $\boxed{\text{div.}}$

③ if $\boxed{\rho = 1} \Rightarrow$ $\boxed{\text{fails}}$ يفشل

Ex: $\sum_{n=1}^{\infty} n^{-n}$

Sol: $a_n = n^{-n} = \frac{1}{n^n}$ By Root Test

$$\boxed{\rho = \lim_{n\to\infty} \sqrt[n]{|a_n|}}$$

$$\rho = \lim_{n\to\infty} \left|\frac{1}{n^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n^{\frac{n}{n}}}$$

$$= \lim_{n\to\infty} \frac{1}{n}$$

$$= \frac{1}{\infty} = 0$$

$\because \rho = 0 < 1 \Rightarrow \sum_{n=1}^{\infty} n^{-n}$ is $\boxed{\text{conv.}}$

T4 Ratio Test اختبار النسبة

Consider the series $\sum a_n$, Let

$$\boxed{\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

① if $\boxed{\rho < 1}$ $\Rightarrow$ $\sum a_n$ $\boxed{\text{conv.}}$

② if $\boxed{\rho > 1}$ $\Rightarrow$ $\sum a_n$ $\boxed{\text{div.}}$

③ if $\boxed{\rho = 1}$ $\Rightarrow$ $\boxed{\text{fails}}$

Ex: $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Sol: $a_n = \frac{n^n}{n!}$ By Ratio Test

$$\boxed{\rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|} = \lim_{n\to\infty} \frac{\frac{(n+1)^{(n+1)}}{(n+1)!}}{\frac{n^n}{n!}}$$

$$\rho = \lim_{n\to\infty} \frac{(n+1)^{(n+1)}}{(n+1)!} * \frac{n!}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)\,(n+1)^n}{(n+1)\,n!} * \frac{n!}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$\because \boxed{\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e}$$

$$\therefore \rho = e > 1 \Rightarrow \sum \frac{n^n}{n!} \text{ is } \boxed{\text{div.}}$$

Factorial : مفكوك

$$n! = n*(n-1)*(n-2)*\cdots*1$$

$0! = 1$, $1! = 1$, $2! = 2*1 = 2$

$3! = 3*2*1 = 3*2! = 6$

$4! = 4*3*2*1 = 4*3! = 4*3*2! = 24$

$$(n+2)! = (n+2)(n+1)! = (n+2)(n+1)\,n!$$

$$n! = n(n-1)! = n(n-1)(n-2)!$$

$$\cdots(n+3)(n+2)(n+1)\,n\,(n-1)(n-2)(n-3)\cdots$$

Exercise: Test for Convergence:

$$\sum_{n=2}^{\infty}\left(\frac{n^3+n-1}{n^2-2n+1}\right)^n$$

$$\sum_{n=0}^{\infty}\frac{n!}{2^n}$$

$$\sum_{n=2}^{\infty}\frac{1}{(\ln(n))^n}$$

$$\sum_{n=0}^{\infty}\frac{3n!}{(2n)!}$$

Lecture 5

$\boxed{T_5}$ Comparison Test اختبار المقارنة

Let $\sum a_n$ and $\sum b_n$ be two series with positive terms $a_n > 0$ and $b_n > 0$

① if $\boxed{\sum b_n \text{ conv.}}$ and $\boxed{b_n \geq a_n}$ $\forall n \geq 1 \Rightarrow \boxed{\sum a_n \text{ conv.}}$

② if $\boxed{\sum b_n \text{ div.}}$ and $\boxed{b_n \leq a_n}$ $\forall n \geq 1 \Rightarrow \boxed{\sum a_n \text{ div.}}$

الأكبر متباعد $\underset{\not\leftarrow}{\rightarrow}$ الأصغر متباعد ، الأصغر متقاربة $\underset{\not\leftarrow}{\rightarrow}$ الأكبر متقاربة

Ex:- $\sum_{n=2}^{\infty} \frac{1}{n-1}$ By Comparison test with $\sum_{n=2}^{\infty} \frac{1}{n}$

$$a_n = \frac{1}{n-1} \overset{=}{\underset{<}{>}} \frac{1}{n} = b_n \Rightarrow a_n > b_n \ \forall n \geq 2$$

$\sum b_n = \sum \frac{1}{n}$ is p-series , $p = 1 \Rightarrow \sum b_n$ div.

$\because \sum b_n = \sum \frac{1}{n}$ div. الأصغر $\Rightarrow \sum a_n = \sum \frac{1}{n-1}$ div. الأكبر

Ex:- $\sum_{n=0}^{\infty} \frac{1}{2^n+2}$ By comparison test with $\sum_{n=0}^{\infty} \frac{1}{2^n}$

$$a_n = \frac{1}{2^n+2} \overset{=}{\underset{>}{<}} \frac{1}{2^n} = b_n \Rightarrow a_n < b_n \ \forall n \geq 1$$

$\sum b_n = \sum \frac{1}{2^n}$ is Geometric Series $|r| = \frac{1}{2} < 1 \Rightarrow \sum b_n$ conv.

$\because \sum b_n = \sum \frac{1}{2^n}$ conv. الأكبر $\Rightarrow \sum a_n = \sum \frac{1}{2^n+2}$ conv. الأصغر

$\boxed{T_6}$ Limit comparison test اختبار المقارنة بالنهاية

Let $\sum a_n$ and $\sum b_n$ be two series with positive terms $a_n > 0$, $b_n > 0$ and let

$$\boxed{\mu = \lim_{n\to\infty} \frac{a_n}{b_n}}$$, then

① if $\boxed{\sum b_n \text{ conv.}}$ and $\boxed{0 \le \mu < \infty}$ $\Rightarrow$ $\boxed{\sum a_n \text{ conv.}}$

② if $\boxed{\sum b_n \text{ div.}}$ and $\boxed{0 < \mu \le \infty}$ $\Rightarrow$ $\boxed{\sum a_n \text{ div.}}$

Ex:- $\sum_{n=1}^{\infty} \frac{1}{n+1}$ By Limit Comp. test with $\sum_{n=1}^{\infty} \frac{1}{n}$

$a_n = \frac{1}{n+1}$, $b_n = \frac{1}{n} \Rightarrow \sum b_n = \sum \frac{1}{n}$ P-series $p=1 \Rightarrow \boxed{\text{div.}}$

$$\boxed{\mu = \lim_{n\to\infty} \frac{a_n}{b_n}} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1} \overset{\frac{d}{dn}}{=} \lim_{n\to\infty} \frac{1}{1} = 1$$

$\because$ $\mu = 1 > 0$ and $\sum b_n$ div. $\Rightarrow \sum a_n = \sum \frac{1}{n+1}$ $\boxed{\text{div.}}$

Ex:- $\sum_{n=2}^{\infty} \frac{1}{2^n - 2}$ By Limit Comp. test with $\sum \frac{1}{2^n}$

$a_n = \frac{1}{2^n - 2}$, $b_n = \frac{1}{2^n} \Rightarrow \sum b_n = \sum \frac{1}{2^n}$ Geometric $|r| = \frac{1}{2} < 1$ $\boxed{\text{conv.}}$

$$\mu = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-2}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{1}{\frac{2^n}{2^n} - \frac{2}{2^n}} = \lim_{n\to\infty} \frac{1}{1 - \frac{1}{2^{n-1}}}$$

$$\therefore \mu = \frac{1}{1 - \frac{1}{2^\infty}} = \frac{1}{1 - \frac{1}{\infty}} = \frac{1}{1-0} = 1 < \infty \Rightarrow \sum a_n = \sum \frac{1}{2^n - 2} \ \boxed{\text{conv.}}$$

$\boxed{T_7}$ Alternating Series test اختبار المتسلسلات التبادلية

The Alternating series are of the forms

$$\sum_{n=1}^{\infty}(-1)^n u_n = -u_1+u_2-u_3\mp\cdots \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n+1} u_n = u_1-u_2+u_3\pm\cdots$$

where $u_n > 0$, if

① $\boxed{u_n > u_{n+1}}$ $\forall n \geq 1$ (decreasing) تناقصية

② $\boxed{\lim_{n\to\infty} u_n = 0}$

Then the alternating series is $\boxed{\text{convergent}}$

Ex:- $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ By alternating Series test

$$a_n = \frac{(-1)^n}{n} = (-1)^n u_n \Rightarrow u_n = |a_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$$

① $u_n = \frac{1}{n} \gtrless \frac{1}{n+1} = u_{n+1}$ (decreasing) ✓

② $\lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = \frac{1}{\infty} = 0$ ✓

$\therefore \sum \frac{(-1)^n}{n}$ is $\boxed{\text{conv.}}$

Absolute convergence and Conditional convergence
التقارب المطلق التقارب المشروط

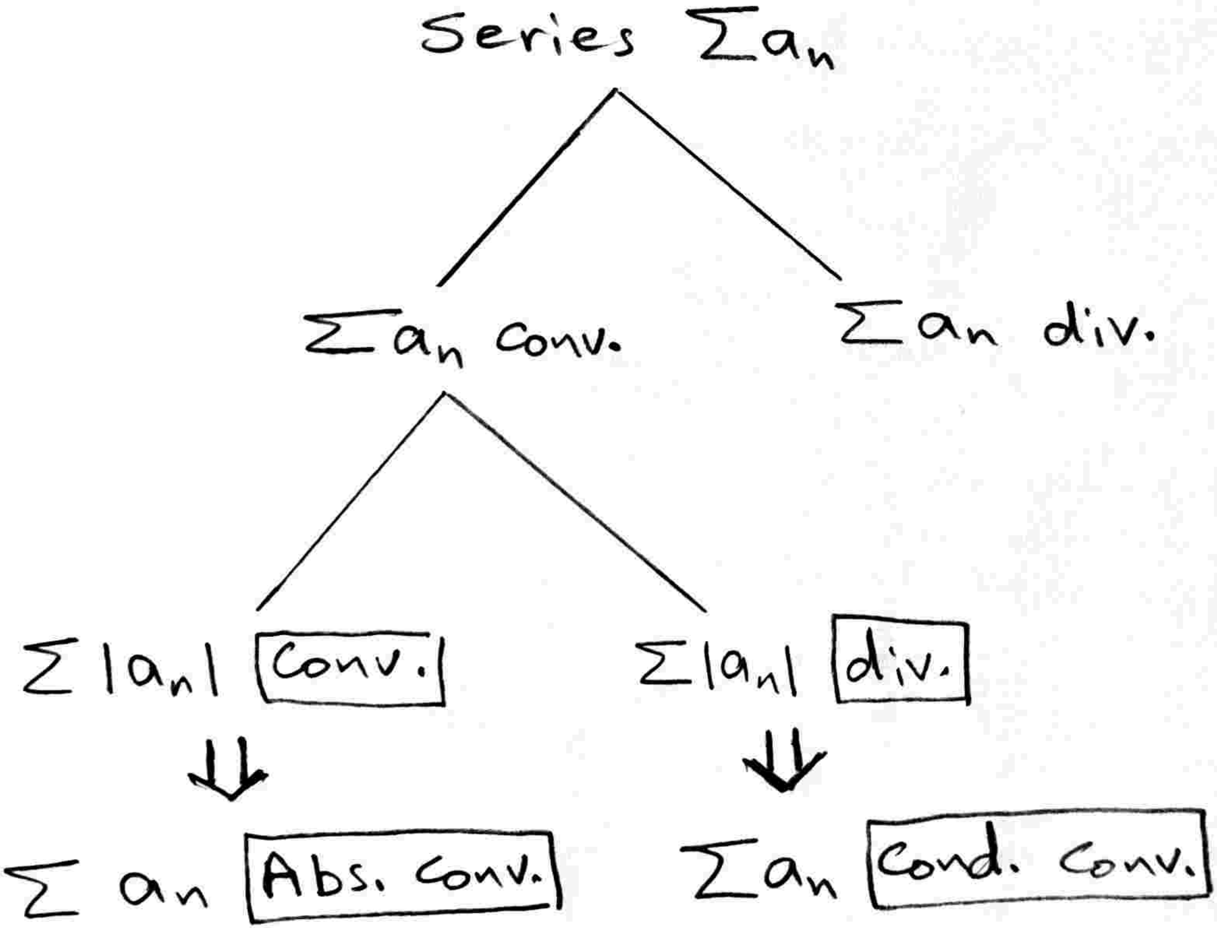
Definition:-

① if $\sum a_n$ conv. and $\sum |a_n|$ $\boxed{\text{conv.}}$
متقاربة بصورة مطلقة

$\Rightarrow$ $\sum a_n$ is Absolutely Conv.

② if $\sum a_n$ conv. and $\sum |a_n|$ $\boxed{\text{div.}}$
متقاربة بصورة مشروطة

$\Rightarrow$ $\sum a_n$ is conditionally conv.



Summary ملخص

① Geometric and p-series

② Divergenc Test

③ Integral Test — إذا كانت a_n قابل للتكامل

④ Root Test — إذا كان $a_n = (\ \ )^n$

⑤ Ratio Test — اذا احتوى a_n على أس n أو $n!$

⑥ Comparison Test — إذا وجدت سلسلة مشابهة بحيث تكون كبيرة متقاربة أو صغيرة متباعدة

⑦ Limit Comp. Test — إذا فشل اختبار المقارنة

⑧ Alternating Series Test — إذا احتوى a_n على $(-1)^n$ أو $(-1)^{n+1}$

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if $\sum \boxed{|a_n| \text{ conv.}} \Rightarrow \sum a_n \boxed{\text{abs. conv.}}$

التقارب في المتسلسلات وحيدة الإشارة (كل الحدود موجبة أو كل الحدود سالبة) يكون تقارباً مطلقاً.

التقارب في المتسلسلات المتناوبة يكون تقارباً مطلقاً إذا كانت $\sum |a_n|$ متقاربة ومشروطاً إذا تباعدت $\sum |a_n|$

Ex:- $\sum \frac{(-1)^n}{n!}$ By ratio test

$a_n = \frac{(-1)^n}{n!}$

$$\rho = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}}{(n+1)!}}{\frac{(-1)^n}{n!}} \right|$$

$$= \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{\cancel{n!}}{(n+1)\cancel{n!}} = \lim_{n\to\infty} \frac{1}{n+1}$$

$\rho = \frac{1}{\infty} = 0 < 1 \Rightarrow \sum \frac{(-1)^n}{n!}$ is Abs. conv.

Exp:- $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ By Alternating Ser. Test

$u_n = |a_n| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}$

① $u_n = \frac{1}{n} \underset{<}{\overset{=}{>}} \frac{1}{n+1} = u_{n+1}$ (decreasing) ✓

② $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$ ✓

$\therefore \sum \frac{(-1)^n}{n}$ conv.

$\because \sum |a_n| = \sum \left|\frac{(-1)^n}{n}\right| = \sum \frac{1}{n}$ is p-series

$P = 1 \Rightarrow \sum \frac{1}{n}$ div.

$\because \sum a_n$ conv. and $\sum |a_n|$ div. $\Rightarrow \sum \frac{(-1)^n}{n}$ is

$\boxed{\text{Cond. conv.}}$

Exercise

Show whether the following series are Abs. conv., Cond. conv. or div.

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1} , \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} , \quad \sum_{n=1}^{\infty} \left(\frac{-1}{2}\right)^n$$

$$\sum_{n=2}^{\infty} \frac{1}{n!-1} , \quad \sum_{n=1}^{\infty} \left(\frac{n^2-1}{n^3-1}\right)^n , \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[3]{n}}$$