Calculus

التفاضل والتكامل

2024-2025

Lecture 1

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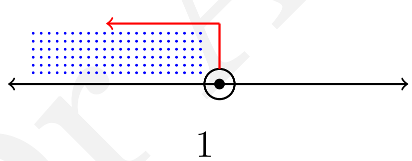
Contents المحتويات: مفردات المادة

Inequalities المتباينات

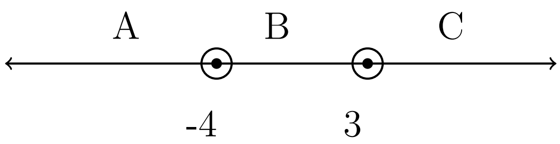
Definition 1. Inequalities: The real numbers can be ordered by size حجم: if $a - b$ is positive, then we write it either $a > b$ or $b < a$.



Example 1. Solve the inequality: $3x + 1 < 5 - x$

$$\begin{aligned} 3x + 1 &< 5 - x \\ 3x + x &< 5 - 1 \\ 4x &< 4 \end{aligned}$$

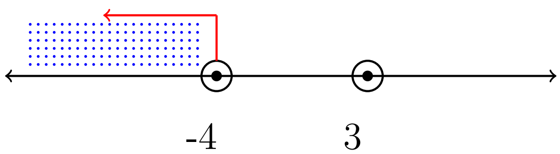
Then, the solution set $S = \{x : x < 1\}$ or $S = (-\infty, 1)$



Example 2. Solve the inequality: $x^2 + x > 12$

$$\begin{aligned} x^2 + x - 12 &> 0 \\ (x+4)(x-3) &> 0 \\ x+4 = 0 &\text{ and } x - 3 = 0 \\ x = -4 &\text{ and } x = 3 \end{aligned}$$

We need to check the correct set of solution S

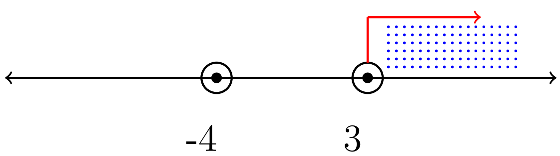


Region A:

take $x = -5$ substitut in $(x+4)(x-3) =$

$-1 \times -8 = 8 > 0$, which is satisfy our condition.

Then, $S_1 = \{x : x < -4\}$

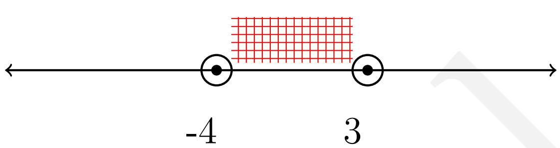


Region C:

take $x = 4$ substitut in $(x+4)(x-3) =$

$8 \times 1 = 8 > 0$, which is satisfy our condition.

Then, $S_2 = \{x : x > 3\}$



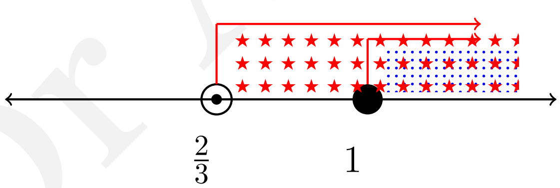
Region B:

take $x = 0$ substitut in $(x+4)(x-3) =$

$4 \times -3 = -12 < 0$, which is not satisfy our condition.

Then, the solution set $S = S_1 \cup S_2 = \{x : x < -4\} \cup \{x : x > 3\}$ or $S = (-\infty, -4) \cup (3, \infty)$.

Example 3. Find the set of solution of $\frac{x}{3x-2} \leq 1$



A +: First, take $3x - 2 > 0$ simplify, $3x > 2$ then $x > \frac{2}{3}$. Now, from the question $\frac{x}{3x-2} \leq 1$, then $x \leq 3x - 2$, which is $x \geq 1$. Then, the intersection between two solutions gives $S_1 = \{x : x \geq 1\}$

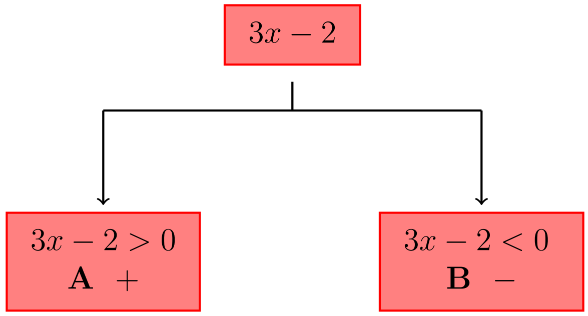
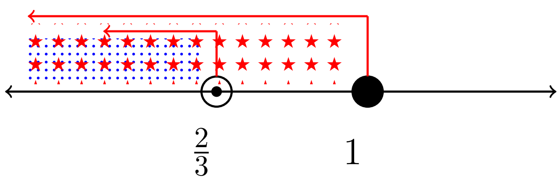


Figure 1: To avoid the denominator المقام not to be zero



B $-$**:** First, take $3x-2<0$ simplify, then $x<\frac{2}{3}$. Now, from the question $\frac{x}{3x-2}\leq 1$, then $x\geq 3x-2$, which is $x\leq 1$. Then, the intersection between two solutions gives $S_2=\{x:x<\frac{2}{3}\}$

Then, the solution set $S=S_1\cup S_2=\{x:x\geq 1\}\cup\{x:x<\frac{2}{3}\}$

Homework

Question 1. *Find the set of solution*

$$1)\ -9<7-3x\leq 12$$
$$2)\qquad \frac{2x+5}{x+4}<1$$

Definition 2. The absolute value or magnitude of a real number $x\in\mathbb{R}$ is defined by:

$$|5|=5,\qquad \left|\frac{-4}{7}\right|=\frac{4}{7}.$$

Properties of Absolute value: if x, a and b are real numbers, then

1. $|-a| = |a|$,
2. $|ab| = |a||b|$,
3. $|a/b| = |a|/|b|, b \neq 0$,
4. $|a+b| \leq |a| + |b|$,
5. $|a| \geq 0$,
6. $|a-b| = |b-a|$,
7. $|a-b| \geq |a| - |b|$,
8. $|x| \leq a, \qquad -a \leq x \leq a$,
9. $|x| > a, \qquad x > a \quad \text{or} \quad x < -a$,
10. $-1 \times (x \geq a) \rightarrow -x \leq -a, \qquad -1(x < a) \rightarrow -x > -a$.

Note 1. If $f(x) = |x|$ and $a \in \mathbb{R}$, then we have two important cases that the question can be:
A) $|f(x)| \leq a$ or $|f(x)| < a$. The set of solution will be $S = \{x : -a \leq x \leq a\}$ or $S = \{x : -a < x < a\}$, respectively.
B) $|f(x)| \geq a$ or $|f(x)| > a$. The set of solution will be $S = \{x : x \geq a\} \cup \{x : x \leq -a\}$ or $S = \{x : x > a\} \cup \{x : x < -a\}$, respectively.

Example 4. Find the set of solution for this inequality: $|x-6| < -3$.

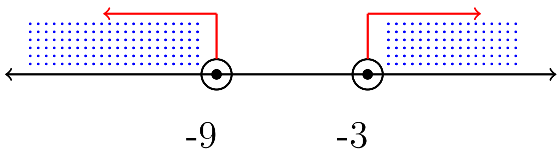
Solution: We will use number 8 in the above note, then

$$3 < x - 6 < -3$$

$$3 + 6 < x - 6 + 6 < -3 + 6$$

$$9 < x < 3$$

then, the set of solution $S = \{x : 9 < x\} \cup \{x : x < 3\}$.



Example 5. Find the set of solution for this inequality: $|x + 6| > 3$.

We will use number 9 in the above note, then

$$x + 6 > 3 \quad \text{or} \quad x + 6 < -3$$
$$x > -3 \quad ** \quad x < -9.$$

Then, the set of solution $S = \{x : x > -3\} \cup \{x : x < -9\}$.

Example 6. Find the set of solution of $\left|\frac{2x+5}{x+4}\right| < 1$.

This example looks like number 8 in the above note, then

$$-1 < \frac{2x+5}{x+4} < 1$$

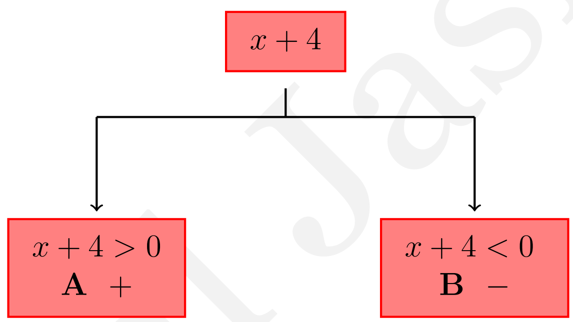


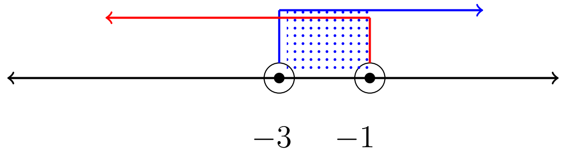
Figure 2: To avoid the denominator المقام not to be zero

A +: First, take $x + 4 > 0 \Rightarrow$, $x > -4$. Now, from the question $-1 < \frac{2x+5}{x+4} < 1$, multiply it by $(x + 4)$ then,

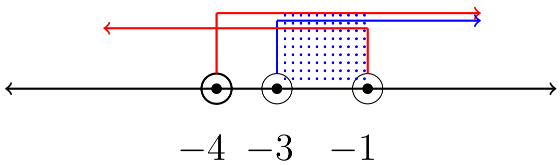
$$-(x + 4) < 2x + 5 < x + 4$$
$$-x - 4 < 2x + 5 < x + 4$$

Find the possible solution

$$-x - 4 < 2x + 5 \quad ** \quad 2x + 5 < x + 4$$
$$2x + x > -4 - 5 \quad ** \quad 2x - x < 4 - 5$$
$$3x > -9 \Rightarrow x > -3 \quad ** \quad x < -1$$



Then, there is an intersection region $\{-3 < x < -1\}$.



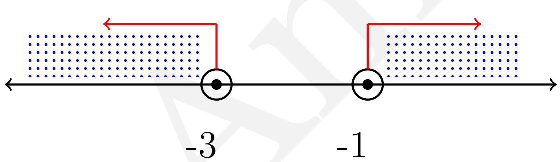
A +: we had from the denomenator $\{x > -4\}$, from the question we had $\{-3 < x < -1\}$. Then, the intersection will be, $S_1 = \{x : -3 < x < -1\}$

B $-$: First, take $x + 4 < 0 \Rightarrow$, $x < -4$. Now, from the question $-1 < \frac{2x+5}{x+4} < 1$, multiply it by $(x + 4)$ then,

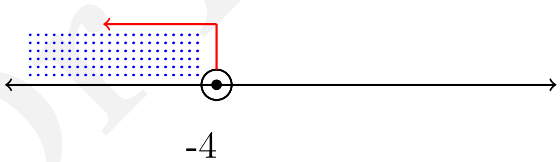
$$-(x + 4) > 2x + 5 > x + 4$$
$$-x - 4 > 2x + 5 > x + 4$$

Find the possible solution

$$-x - 4 > 2x + 5 \quad ** \quad 2x + 5 > x + 4$$
$$2x + x < -4 - 5 \quad ** \quad 2x - x > 4 - 5$$
$$3x > -9 \Rightarrow x < -3 \quad ** \quad x > -1$$



Then there is no intersection, Which means $\{x : x < -3\} \cap \{x : x > -1\} = \phi$



B $-$**:** we had from the denomenator $\{x < -4\}$, from the question we had ϕ. Then the intersection between them $S_2 = \phi \bigcap \{x < -4\} = \phi$

Finally, the set of solution $S = S_1 \cup S_2 = \{x : -3 < x < -1\}$.

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Lecture 2

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Functions الدوال

Definition 1. Function: If a variable y depends on a variable x in such a way that each value of x determines exactly one value of y is a function of x. So, If $f(x) = y$, then the set of all possible input (x- values) is called the domain of f (D_f) and the set of output (y- values) is called the range of f (R_f). We can write the mapping as:

$$f : \text{domain} \longrightarrow \text{co-domain.}$$

Present Different Type of Functions Graphing and Analysing their Domain and Range

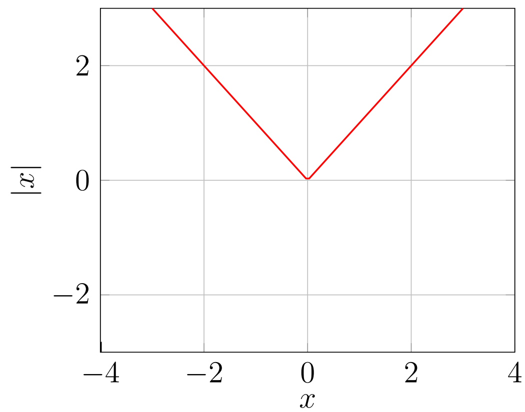
1) The absolute value function المجال والمدى لدالة القيمة المطلقة

The graph of the function $f(x) = |x|$ can be obtained by the two parts of equation:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0. \end{cases}$$

x	$f(x) = \lvert x\rvert$
$\vdots$	
-2	$-(-2) = 2$
-1	$-(-1) = 1$
0	0
2	2
1	1
$\vdots$	

$D_f = \mathbb{R},\ R_f = \{y : y \geq 0\}$

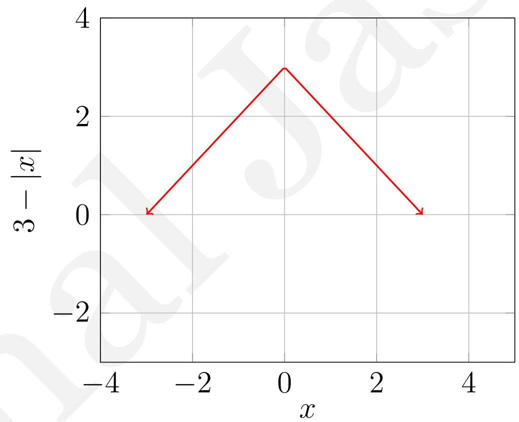


Example 1. Graph the equation and find the D_f and R_f $y = f(x) = 3 - |x|$

$$3 - |x| = \begin{cases} 3 - x & x \geq 0 \\ 3 + x & x < 0. \end{cases}$$

x	$f(x) = 3 - \lvert x\rvert$
$\vdots$	
-3	0
-1	2
0	3
2	1
1	2
$\vdots$	

$D_f = \mathbb{R},\ R_f = \{y : y \leq 3\}$



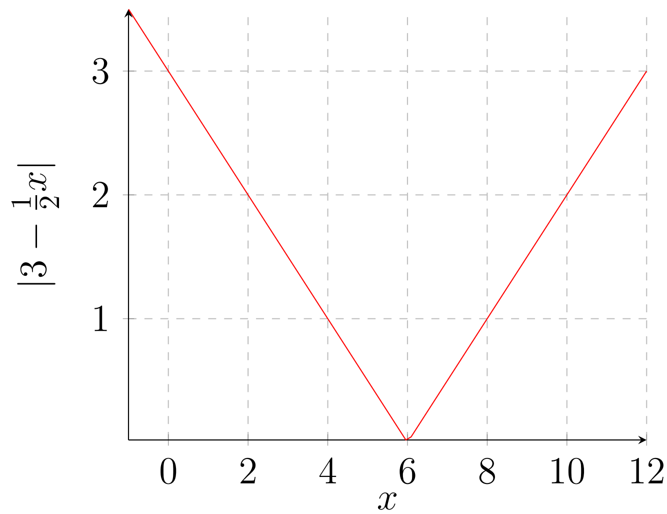
Example 2. Graph the equation and find the D_f and R_f $y = f(x) = |3 - \frac{1}{2}x|$

$$|3 - \frac{1}{2}x| = \begin{cases} 3 - \frac{1}{2}x & 3 - \frac{1}{2}x \geq 0 \\ \frac{1}{2}x - 3 & 3 - \frac{1}{2}x < 0. \end{cases}$$

$$|3 - \frac{1}{2}x| = \begin{cases} 3 - \frac{1}{2}x & x \leq 6 \\ \frac{1}{2}x - 3 & x > 6. \end{cases}$$

x	$f(x) = \|3 - \frac{1}{2}x\|$
$\vdots$	
0	3
2	2
4	1
6	0
8	1
10	2
$\vdots$	

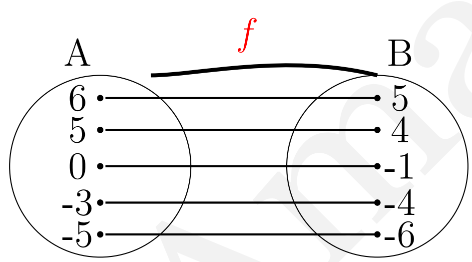
$D_f = \mathbb{R},\ R_f = \{y : y \geq 0\}$



Definition 2. Surjective mapping: if the range = the co-domain.

Definition 3. Injective mapping: If each element in B (range) connected with only one element in A (domain).

Definition 4. Bijective mapping: If the mapping is surjective and injective at the same time.



Example 3. Example of a function:

$f : A \to B, \quad$ where $f(x) = y = x - 1$

$A = \{6, 5, 0, -3, -5\}$

$B = \{5, 4, -1, -4, -6\}$,

$D_f = A, \quad R_f = B.$

f is surjective and injective. Then, f is bijective.

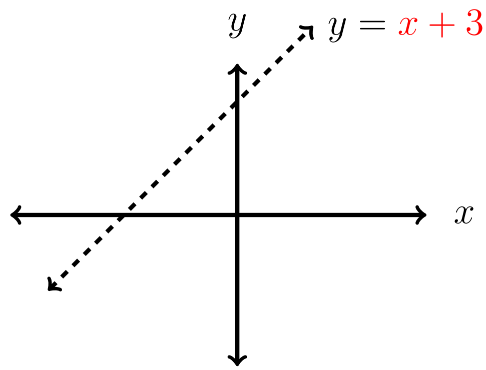
Example 4. Find the D_f and R_f to this function

$$y = f(x) = x + 3,$$

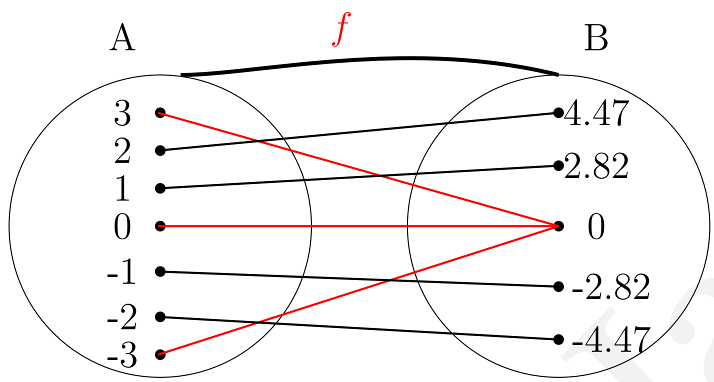
where $f : \mathbb{R} \to \mathbb{R}$. In this case $D_f = \mathbb{R}$ and the co-domain is $\mathbb{R}$.

x	$f(x) = x + 3$
$\vdots$	$\vdots$
0	3
1	4
-1	2
-2	1
$\vdots$	$\vdots$

f : surjective, injective, bijective.



$D_f = \mathbb{R},\ R_f = \mathbb{R}$



Example 5. Example of a function:

$f : A \to B, \quad f(x) = y = x\sqrt{9 - x^2},$

$x \in -3 \le x \le 3 = A, \quad y \in B.$

f is surjective, not injective. Then, f it is not bijective.

Example 6. Find the D_f and R_f to this function

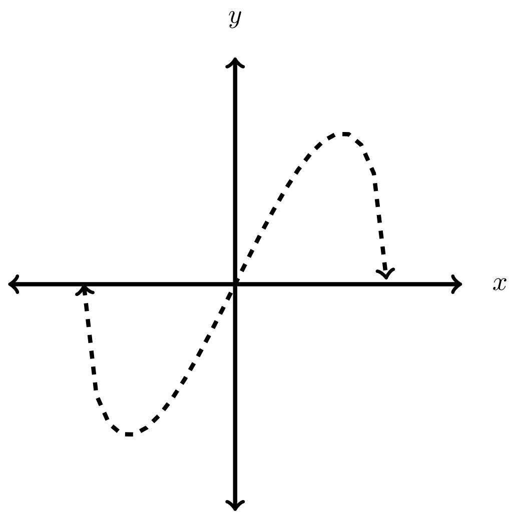
$$y = f(x) = x\sqrt{4 - x^2},$$

where $f : \{-2 \le x \le 2\} \to \mathbb{R}$. In this case $D_f = \{-2 \le x \le 2\}$ and the co-domain is $\mathbb{R}$.

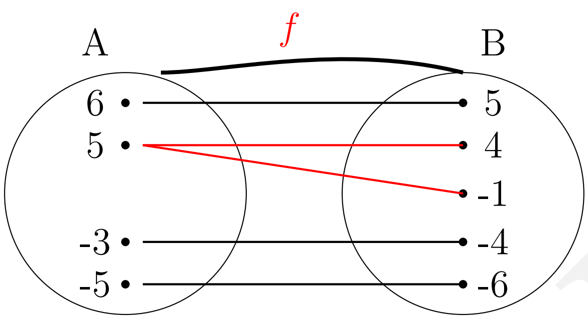
x	$f(x) = x\sqrt{4-x^2}$
2	0
$\sqrt{2}$	2
1	1.73
0	0
-1	-1.73
-2	0

not(surjective, injective, bijective).

symmetric arround the origin



$D_f = \{-2 \le x \le 2\}, \; R_f = \{-2 \le y \le 2\}$



Example 7. Example of f which is not a function:

There are two points $y = 4, -1 \in B$ which are related to a one point $x = 5 \in A$.

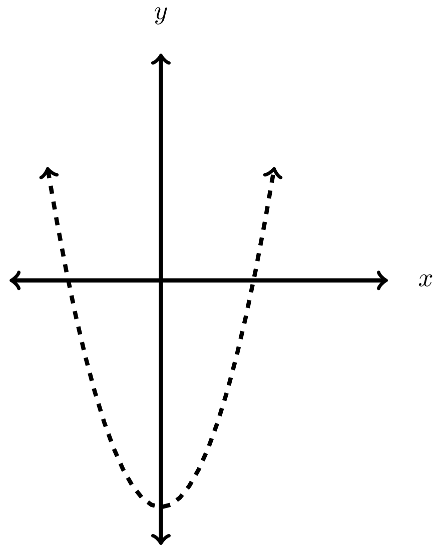
Homework

Question 1. *What is the type of the mapping, where Z is represent the set of integer numbers? $f : Z \to Z$, where $f(x) = 2x^2 - 3$.*

Solution 1. $D_f = Z$ and co-domain$= Z$. Then, f : is not surjective, not injective. Then it is not bijective.

x	$f(x) = 2x^2 - 3$
$\vdots$	$\vdots$
2	5
1	-1
0	-3
-1	-1
-2	5
$\vdots$	$\vdots$

$D_f = Z,\ R_f = \{y \geq -3\}$

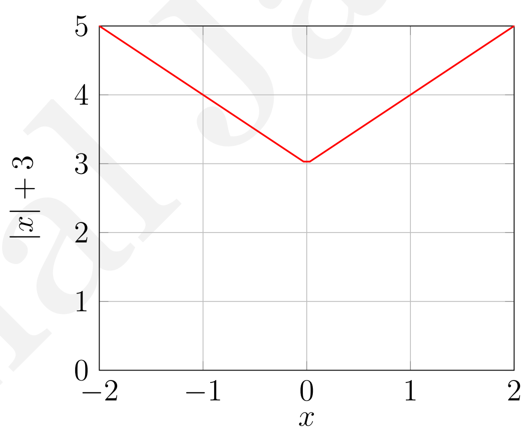


Example 8. Graph the equation and find the D_f and R_f of this function $f(x) = y = |x| + 3$.

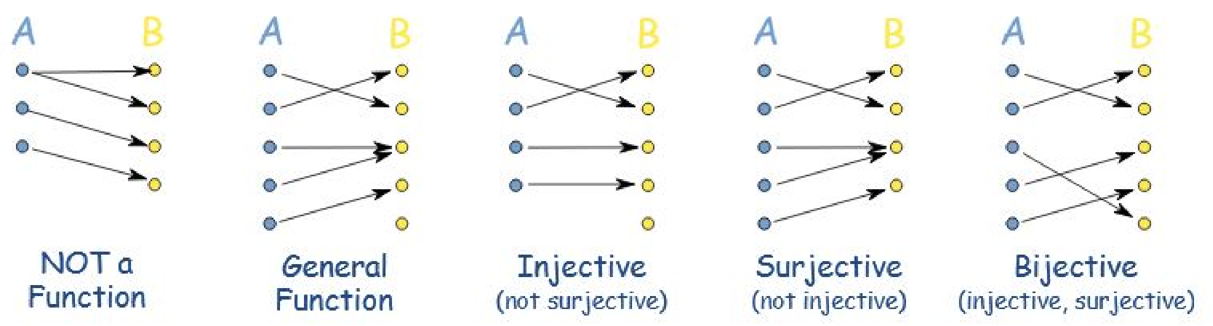
$$|x| + 3 = \begin{cases} x + 3 & x \geq 0 \\ -x + 3 & x < 0. \end{cases}$$

x	$f(x) = \|x\| + 3$
$\vdots$	
-2	5
-1	4
0	3
1	4
2	5
$\vdots$	

$D_f = \mathbb{R},\ R_f = \{y : y \geq 3\}$



Note: **A** is domain, **B** is co-domain in the following figures.



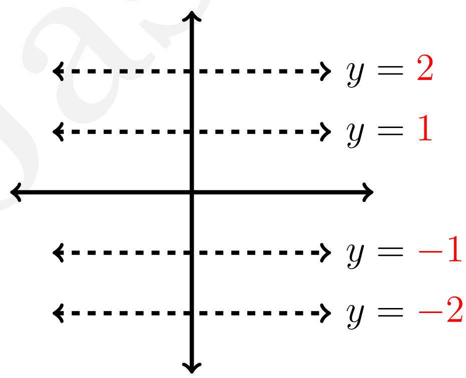
Injective, Surjective and Bijective : tells us about how a function behaves.

2) Constant function المجال والمدى للدالة الثابته

Definition 5. Constant function: A function f whose values are all the same.

Example: Find the domain and the range for $f(x) = -2$, in this function the range R_f is always -2 for all $x \in D_f$.

x	$f(x) = -2$
-2	-2
-1	-2
0	-2
2	-2



$D_f = \mathbb{R}, \; R_f = \{y : y = -2\}$

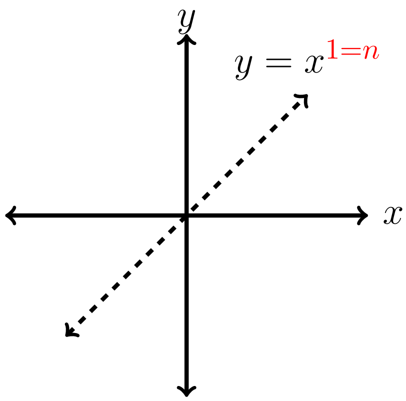
3) A power function دالة القوى

Definition 6. A power function : A function of the form $f(x) = x^p$, where p is constant.

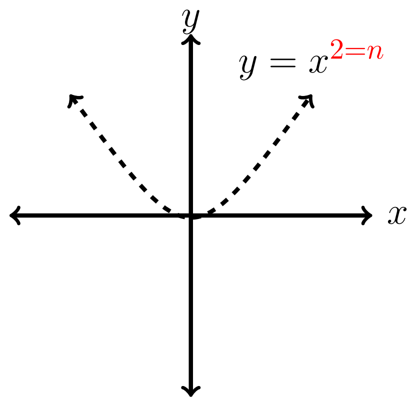
We can represent the power function as $f(x) = x^p$. In this case we have three kind of p: when $p > 0, p < 0$ and p is a fraction كسر.

3-A) دالة القوى عندما القوى موجبة

If we say $p = n$ and n is a positive integer.



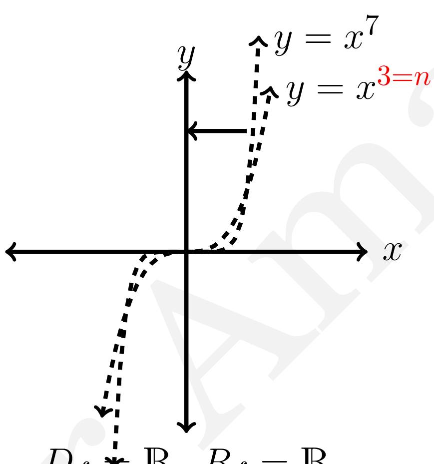
$D_f = \mathbb{R},\ R_f = \mathbb{R}$



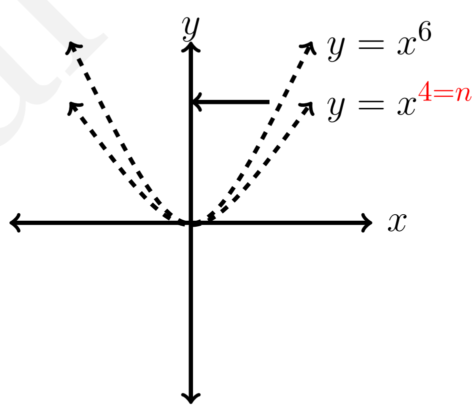
$D_f = \mathbb{R},\ R_f = \{y : y \geq 0\}$

x	$f(x) = x$
-2	-2
-1	-1
0	0
2	2

x	$f(x) = x^2$
-2	4
-1	1
0	0
2	4



$D_f = \mathbb{R},\ R_f = \mathbb{R}$



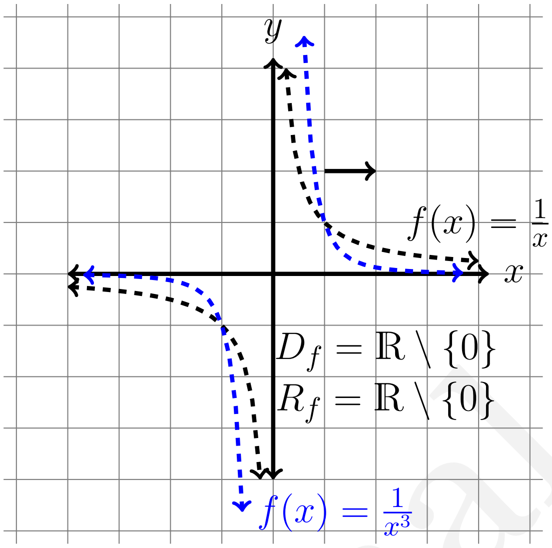
$D_f = \mathbb{R},\ R_f = \{y : y \geq 0\}$

Note 1. For even value of n, the function $f(x) = x^n$ are even, and their graphs are symmetric about the $y-axis$. For odd values of n the functions $f(x) = x^n$, are odd and their graphs are symmetric about the origin, that means about the point $(0, 0)$.

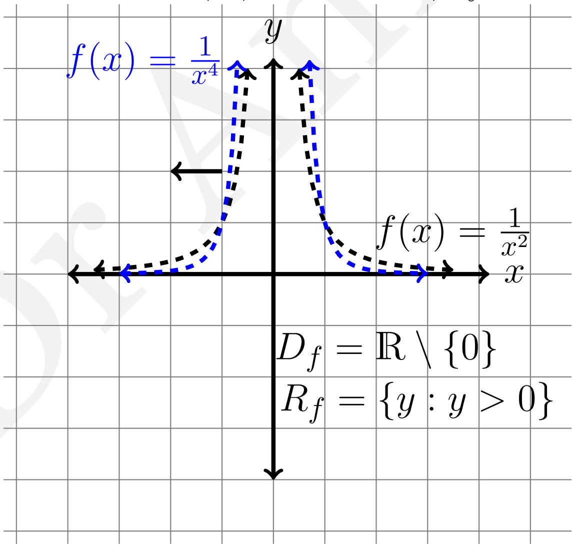
3-B) دالة القوى عندما القوى سالبة

If we say $p = n$ and p is a negative integer. Say $p = -n$, n is integer, then $f(x) = x^{-n} = \frac{1}{x^n}$.

When $n = 1, 3, \ldots$ is odd, symmetric about the origin $(0, 0)$.



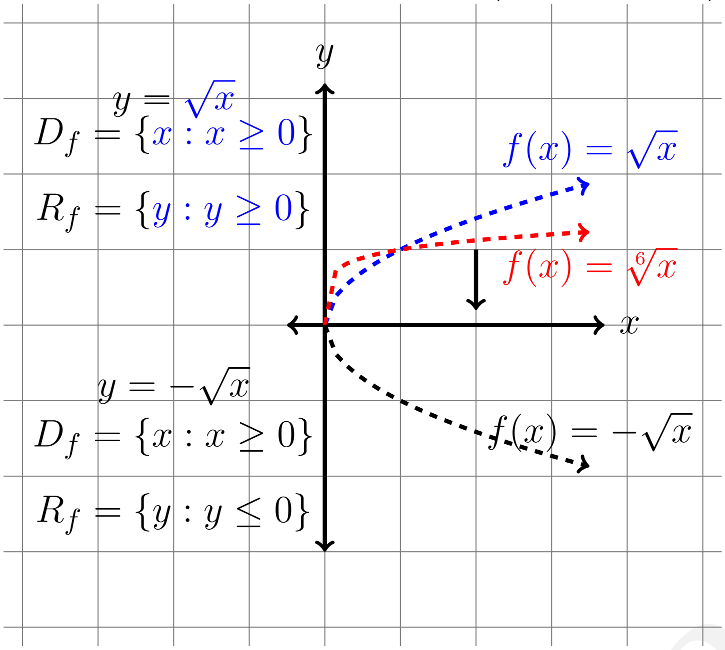
When $n = 2, 4, \ldots$ is even, symmetric about the $y - axis$.



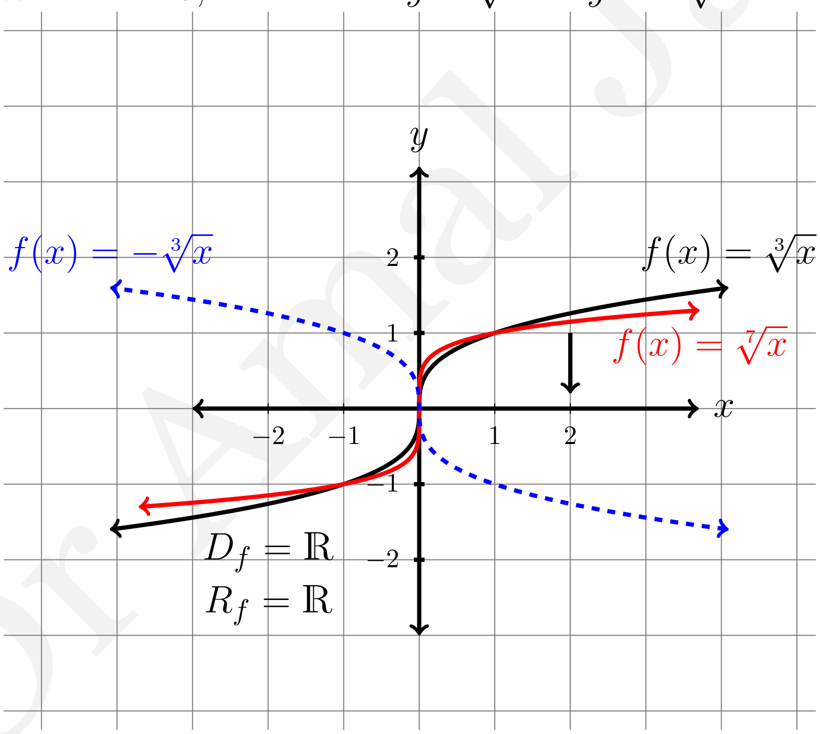
3-C) دالة القوى عندما القوى عبارة عن كسر

If we say $p = \frac{1}{n}$, n is positive integer. Then $f(x) = x^p = x^{\frac{1}{n}} = \sqrt[n]{x}$.

When $n = 2$, this means $y = \sqrt{x}$ or $y = -\sqrt{x}$:



When $n = 3$, this means $y = \sqrt[3]{x}$ or $y = -\sqrt[3]{x}$:



Calculus

التفاضل والتكامل

2024-2025

Lecture 3

Ass. Prof. Dr. Amal Jasim Mohammed

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4) Polynomial function متعددة حدود

Definition 1. The polynomial function has this form:$y = f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$. $a_0, a_1 \ldots a_n$ are constants and n is non-negative integer.

$$\begin{aligned} 3+5x &\quad \text{has degree1 (linear)},\\ x^2-3x+1 &\quad \text{has degree2 (quadratic)},\\ 2x^3-7 &\quad \text{has degree3 (cubic)}\\ 8x^4-9x^3+5x-3 &\quad \text{has degree4 (quartic)} \end{aligned}$$

Note 1. مهمة

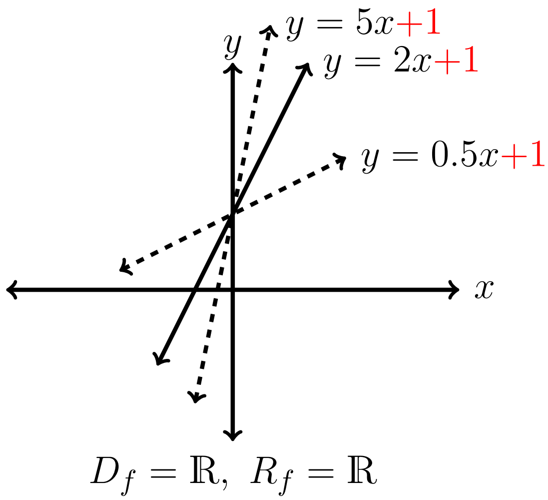
When the roots of a function in $\mathbb{R}$, then

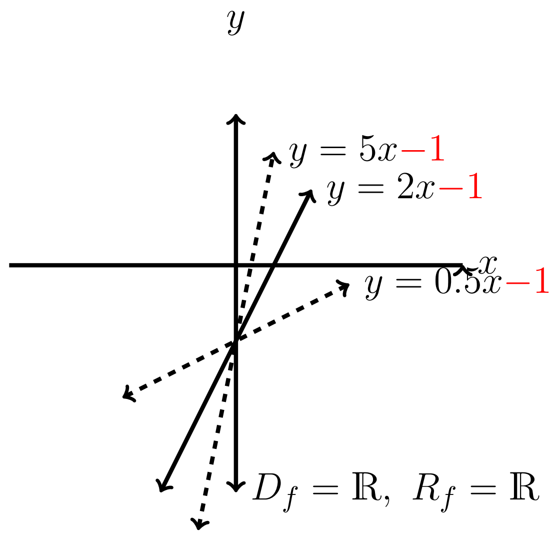
- Linear function (degree 1) cross the x-axis on a point.
- Quadratic function (degree 2) cross the x-axis on two points.
- Cubic function (degree 3) cross the x-axis on three points.
- Quartic function (degree 4) cross the x-axis on four points.

4-A) Linear function

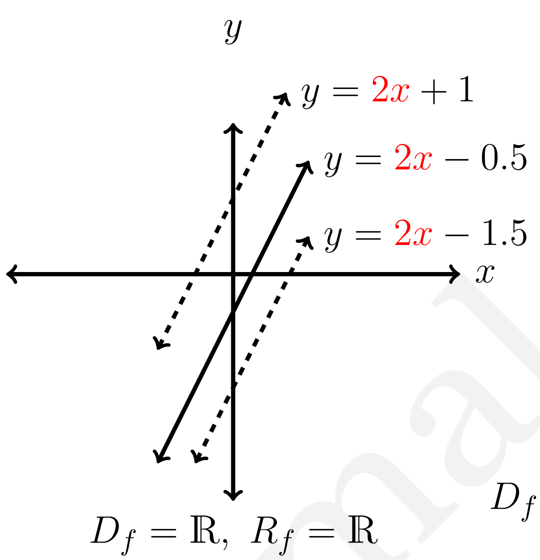
The general form of a linear equation is $y = mx + b$, where $m, b \in \mathbb{R}, m \neq 0$.

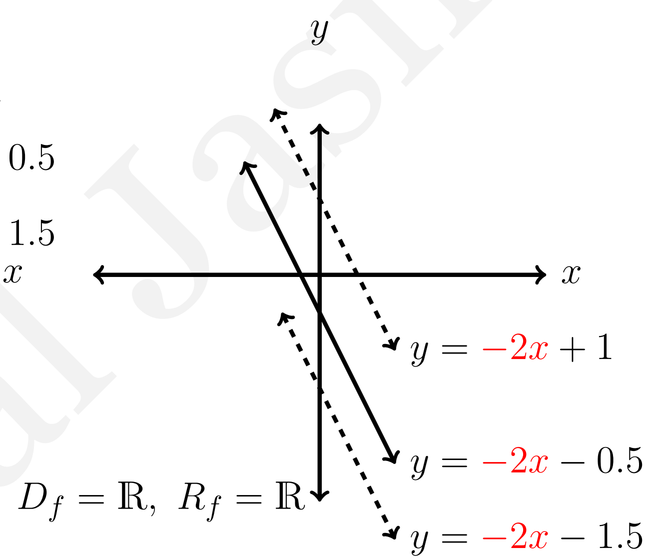
If we keep b fixed and vary مختلفة the parameter m in the equation $y = mx + b$, then we obtain a family of lines. b is the point of intersection.





If we keep m fixed and vary مختلفة the parameter b, then we obtain a family of parallel line and they all have same slope m.





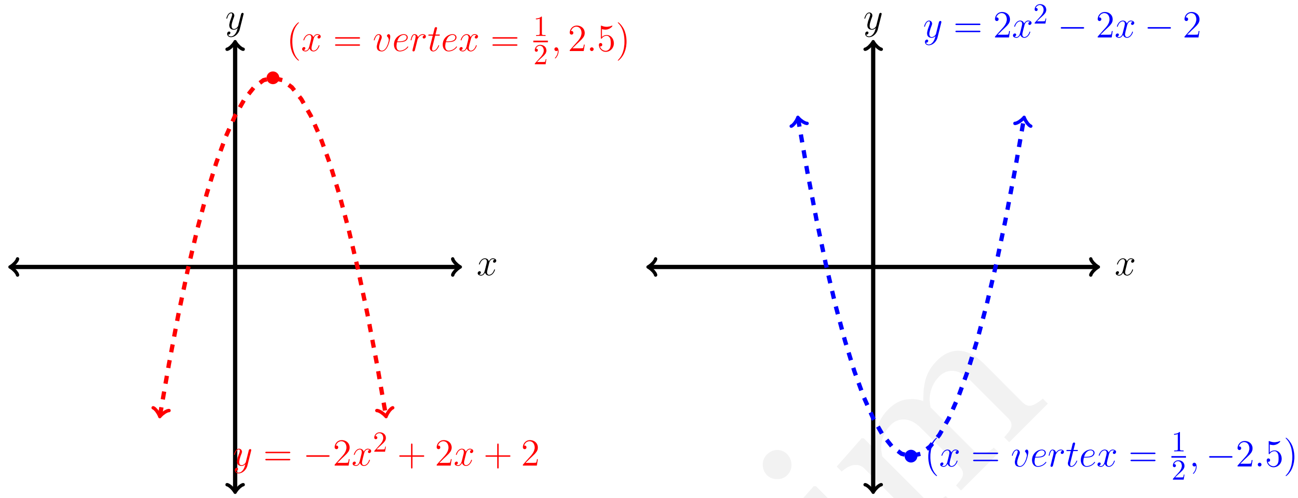
Homework

Question 1. We have $y(x) = \frac{1}{2}x - 1$.

- Find a parallel line for y and find the D_f, R_f.
- Find a line that cross يقطع y, and share يشارك it with a point.

4-B) Quadratic function

An equation of the form $y = ax^2 + bx + c, a \neq 0$, where $b, c \in \mathbb{R}$, is called a quadratic equation in x. Depending on a is positive or negative.



$D_f = \mathbb{R},\ R_f = \{y : y \leq 2.5\}$ $D_f = \mathbb{R},\ R_f = \{y : y \geq -2.5\}$

To find the R_f substitute $x =$ vertex into the y function to find the $y(x = \text{vertex} = \frac{1}{2})$.

Note 2. مهمة

To find the vertex of $y = ax^2 + bx + c : x = \frac{-b}{2 \times a}$.

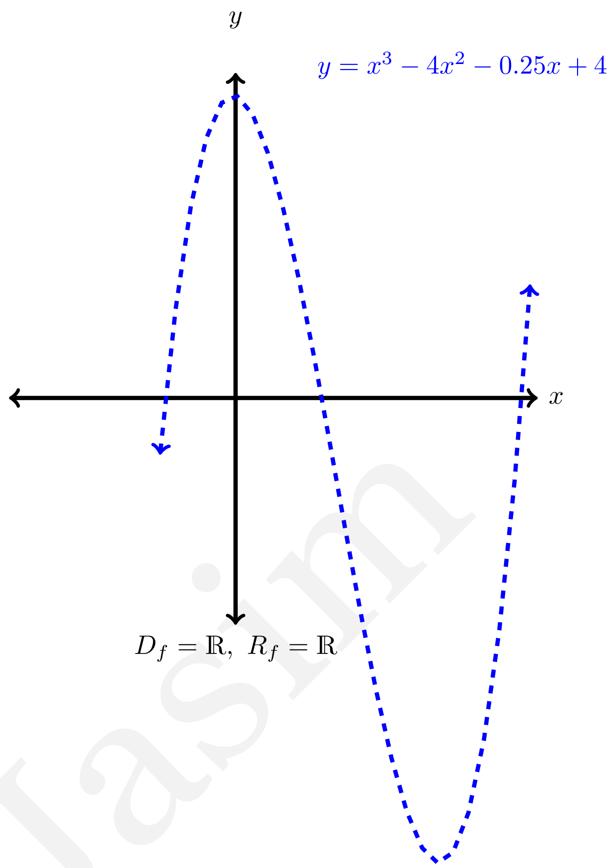
Homework

Question 2. Sketch the graph and find the D_f and R_f and find the vertex of y.

1) $y = x^2 - 2x - 2$

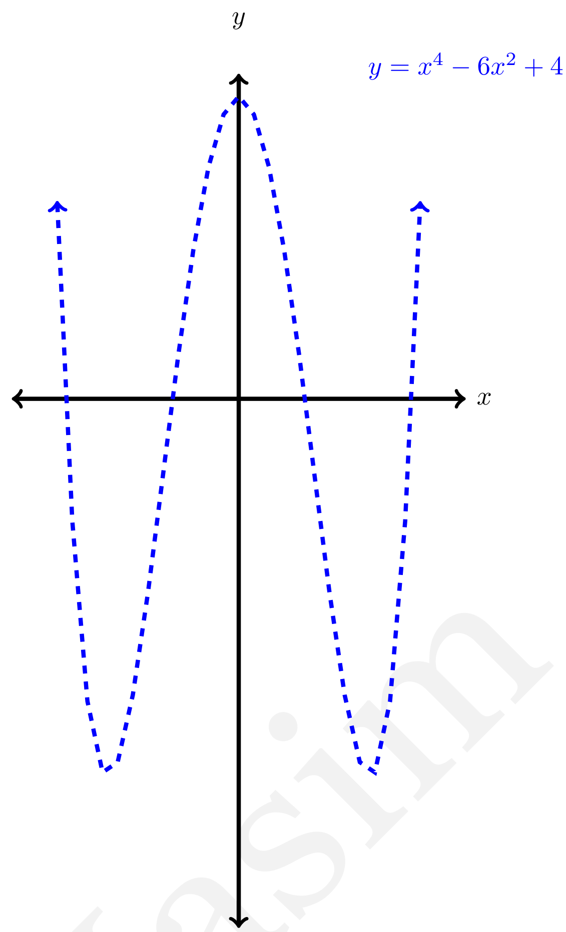
2) $y = x^2 - 2x + 1$

4-C) Cubic function للاطلاع



x	$y = x^3 - 4x^2 - 0.25x + 4$
-2	-19.50
-1	-0.75
0	4
1	0.75
2	-4.50

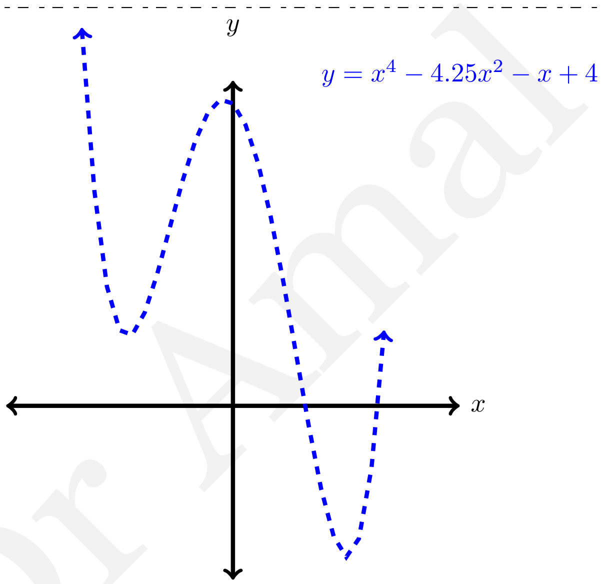
4-D) Quartic function للاطلاع



x	$y = x^4 - 6x^2 + 4$
-2	-4
-1	-1
0	4
1	-1
2	-4

$D_f = \mathbb{R},\ R_f = \{y : y \geq -5\}$

The roots are $-2.288245611, -.8740320489, .8740320489, 2.288245611$.



$D_f = \mathbb{R},\ R_f = \{y : y \geq -2\}$

Example of a quartic function:
has two real roots and two imagenary roots

9) Rational function المجال والمدى للدالة الكسرية

Definition 2. A function that expressed as a ratio of two polynomials called a rational function.

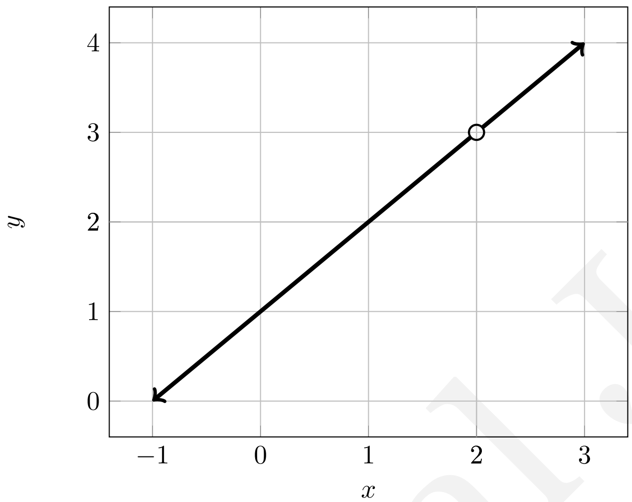
If $P(x)$ and $Q(x)$ are polynomials, then the domain of $\frac{P(x)}{Q(x)}$ is defined as in the arithmetic operation on functions (3).

Example 1. Find the domain and range for this function: $f(x) = \frac{x^2-x-2}{x-2}$.

Since $f(x)$ is rational then, we need to avoid the zero of the denominator : $x = 2$.

$$f(x) = \frac{(x-2)(x+1)}{x-2}$$
$$f(x) = y = x+1$$

The graph of $y = x + 1$ is a linear function but, the graph will have a hole فجوة:شكل دائرة فارغة at $x = 2, y = 3$. At the same time, if we change the function $x \to y$, this will be $x = y - 1$ has no problem.



$D_f = \mathbb{R} \setminus \{2\}$
and $R_f = \mathbb{R} \setminus \{3\}$

Homework

Question 3. Sketch the graph and find the domain and range for this function:

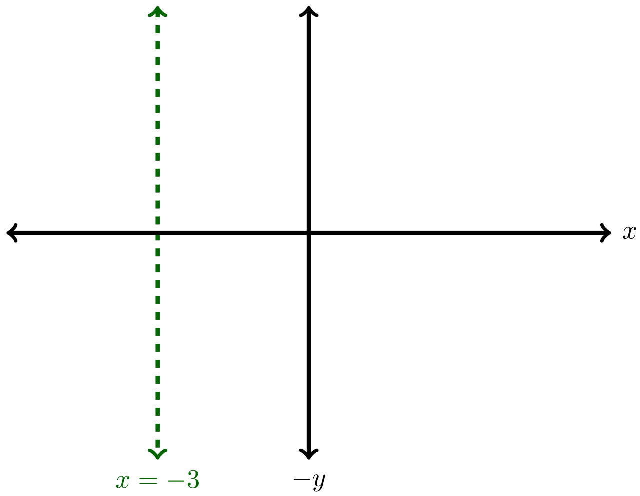
$$y = \frac{x^3-8}{x-2}$$

note: use the vertex.

Example 2. Find the domain and range for this function: $f(x) = \frac{1}{x+3} - 5$.

First, simplify:

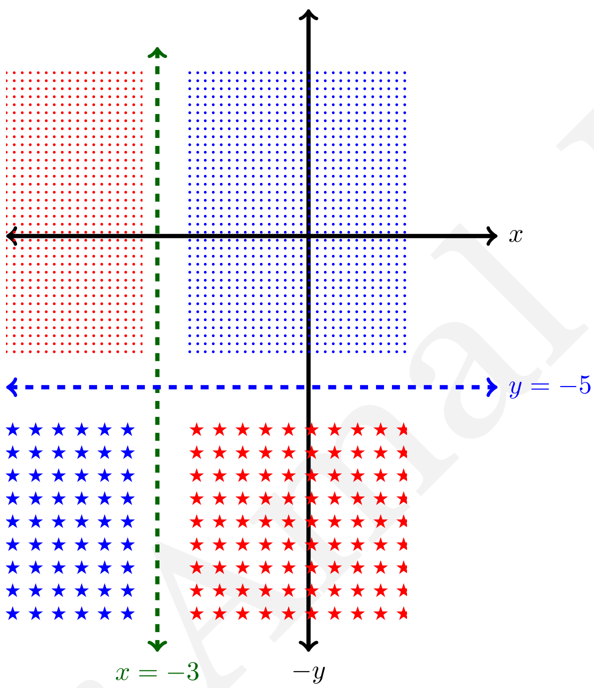
$$f(x) = \frac{1-5(x+3)}{x+3},$$
$$y = \frac{-14-5x}{x+3}.$$



In this case, $D_f = \mathbb{R} \setminus \{-3\}$.
So we have a vertical line $x = -3$.

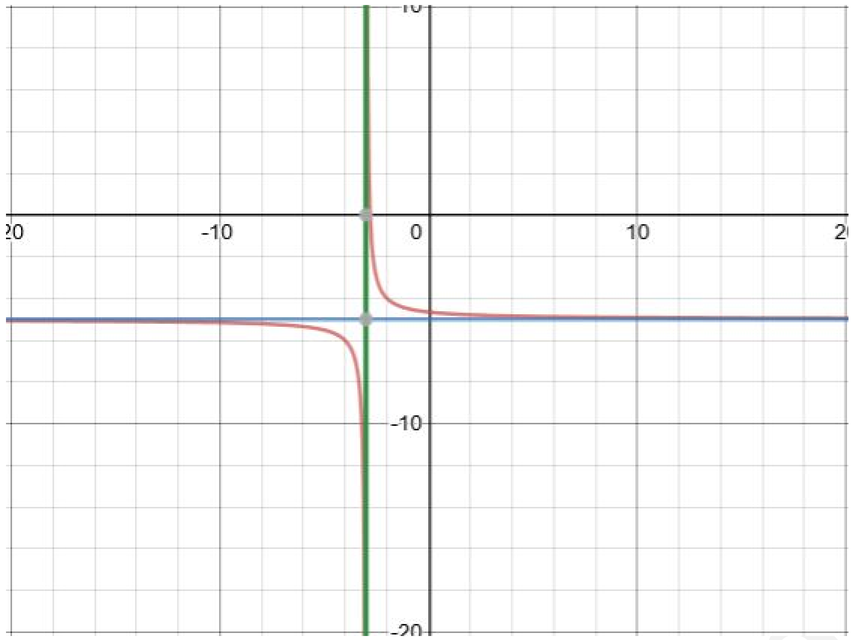
If we change the function $x \to y$, this will be

$$\begin{aligned} y(x+3) &= -14 - 5x, \\ yx + 3y &= -14 - 5x, \\ yx + 5x &= -14 - 3y, \\ x(y+5) &= -14 - 3y, \\ x &= \pm \frac{-14-3y}{y+5}. \end{aligned}$$



In this case, $y = -5$
is a problem.

نتوقع بان يكون الرسم في
المناطق الاربعة (ليس شرط في جميعها)
حسب قيم x



$$D_f = \mathbb{R} \setminus \{x = -3\} \text{ and } R_f = \mathbb{R} \setminus \{y = -5\}$$

Homework

Question 4. Find the domain and the range for the following functions:

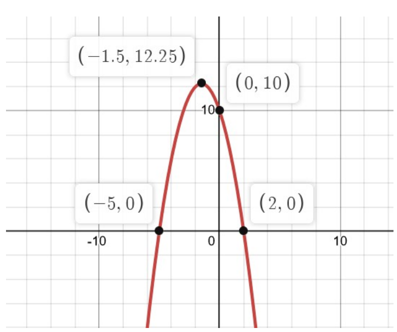
$$f(x) = \frac{x^2 - 3x - 4}{x + 1}, \qquad f(x) = \frac{x^2}{x^2 - 2}$$

More Prosperities of Quadratic تربيعية and Rational نسبية Functions.

Example 3. Sketch and find the Domain and the Range of the following function:

$$f(x) = 10 - 3x - x^2$$

1) y-intersection: $f(0) = 10,\ (0, 10)$.
2) x-intersection: $10 - 3x - x^2 = 0 \to (5 + x)(2 - x) = 0$, so $x = -5$ and $x = 2$, then $(-5, 0),\ (2, 0)$.
3) the vertex y is $x = \frac{-b}{2a} = \frac{-(-3)}{2(-1)} = -1.5$, then, $f(-1.5) = 10 - 3(-1.5) - (-1, 5)^2 = 12.25$, (-1.5,12.25)(convex).



$$D_f = \mathbb{R} \text{ and } R_f = \{y : y \leq 12.25\}$$

Homework

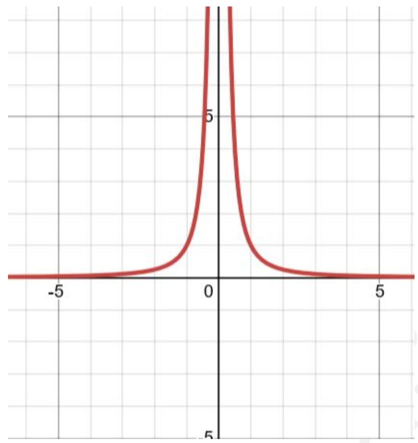
Question 5. Find the domain and the range for the following functions:

$$f(x) = 20 - 3x - x^2, \qquad f(x) = x^2 + 4x + 3$$

Example 4. Sketch and find the Domain and the Range and find the symmetric point (or line) of the following function:

$$f(x) = \frac{1}{x^2}$$

The symmetric line is $y - axis$.



$$D_f = \mathbb{R} \setminus \{0\} \text{ and } R_f = \{y : y > 0\}$$

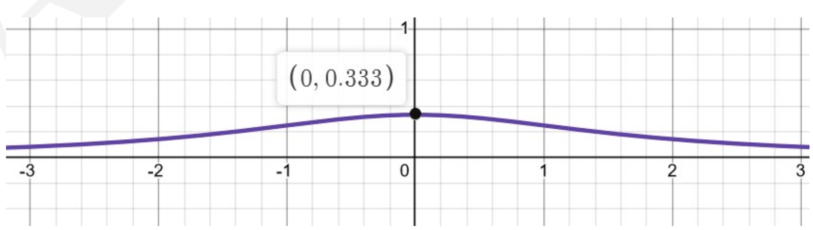
Example 5. Sketch and find the Domain and the Range and find the symmetric point (or line) of the following function:

$$f(x) = \frac{1}{x^2 + 3}$$

1) y-intersection: $f(0) = \frac{1}{3} = 0.333$, $(0, 0.333)$.

$$\begin{aligned} yx^2 + 3y &= 1 \\ x^2 &= 1 - 3y \\ x = \pm &\sqrt{\frac{1-3y}{y}}, \end{aligned}$$

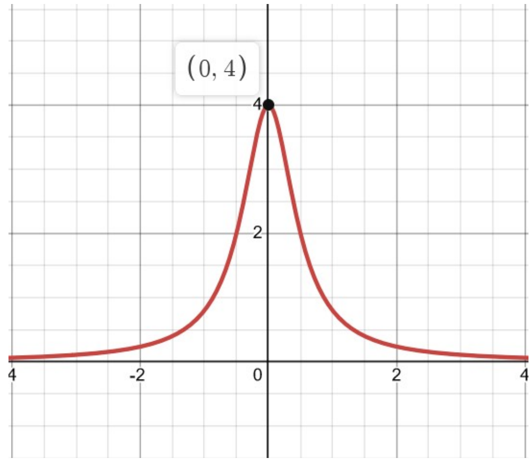
which means, $D_x = \{y : y \leq 0\}$ makes problem. كل عناصر $y \leq 0$ تسبب مشكلة



$$D_f = \mathbb{R} \text{ and } R_f = \{y : 0 < y \leq 0.3333\}$$

Example 6. Sketch and find the Domain and the Range and find the symmetric point (or line) of the following function:

$$f(x) = \frac{1}{x^2 + 0.25}$$



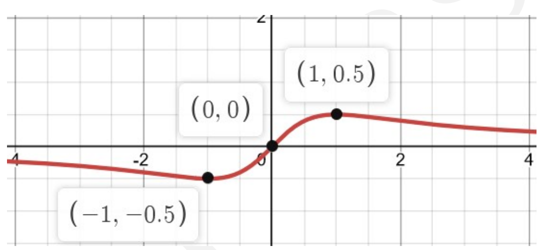
$$D_f = \mathbb{R} \text{ and } R_f = \{y : 0 < y \leq 4\}$$

1) y-intersection: $f(0) = \frac{1}{0.24} = 4$, $(0, 4)$

Note 3. When the numerator البسط has lower degree than the denominatorالمقام, we can not change $x \to y$.

$$f(x) = \frac{x}{x^2 + 1}$$

1) y-intersection: $f(0) = 0$, $(0, 0)$.
2) function y symmetric around $(0, 0)$.



$$D_f = \mathbb{R} \text{ and } R_f = \{y : -0.5 \leq y \leq 0.5\}$$

Note 4. When the numerator and the denominator have the same degree.

Example 7. Sketch and find the Domain and the Range and find the symmetric point (or line) of the following function:

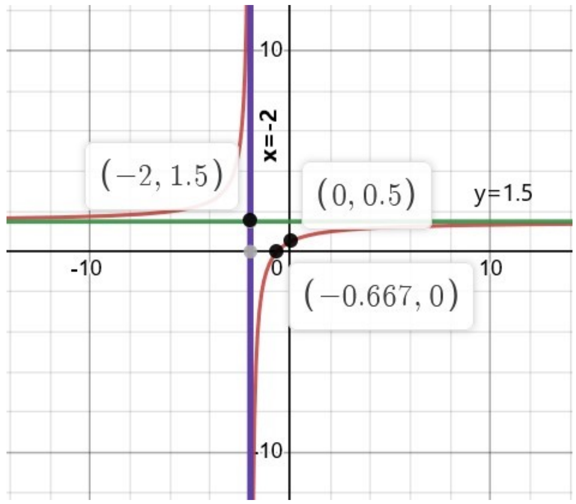
$$f(x) = \frac{3x + 2}{2x + 4}$$

In this case, $D_f = \mathbb{R} \setminus \{-2\}$, so we have a vertical line $x = -2$. To find the R_f, we need to change the function $x \to y$, this will be

$$\begin{aligned} y(2x + 4) =& 3x + 2 \\ 2xy + 4y =& 3x + 2 \\ 2xy - 3x =& -4y + 2 \\ x =& \frac{2 - 4y}{2y - 3} \end{aligned}$$

In this case $y = \frac{3}{2}$ is a problem, which means, we have a horizontal line $y = 1.5$.

1) y-intersection: $f(0) = \frac{2}{4} = 0.5$, $(0, 0.5)$.
2) x-intersection: $\frac{3x+2}{2x+4} = 0 \to 3x + 2 = 0 \to x = -\frac{2}{3} = -0.667$, $(-0.667, 0)$.
3) symmetric point: $(-2, 1.5)$.



$$D_f = \mathbb{R} \setminus \{-2\} \text{ and } R_f = \mathbb{R} \setminus \{1.5\}$$

Example 8. Sketch and find the Domain and the Range and find the symmetric point (or line) of the following function:

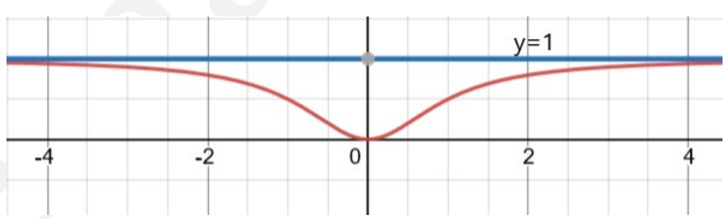
$$f(x) = \frac{x^2}{x^2+1}$$

1) y-intersection: $f(0) = 0,\ (0,0) = x$-intersection.

If we change the function $x \to y$, this will be

$$\begin{aligned} y =& \frac{x^2}{x^2+1}, \\ y(x^2+1) =& x^2, \\ yx^2 + y =& x^2, \\ x^2(y-1) =& -y, \\ x =& \pm\sqrt{\frac{y}{1-y}}. \end{aligned}$$

Which means $D_x = \{y : y \geq 1\}$ and $D_x = \{y : y < 0\}$ make problems.



$$D_f = \mathbb{R} \text{ and } R_f = \{y : 0 \leq y < 1\}$$

Example 9. Sketch and find the Domain and the Range of the following function:

$$f(x) = \frac{1}{x^2-1}$$

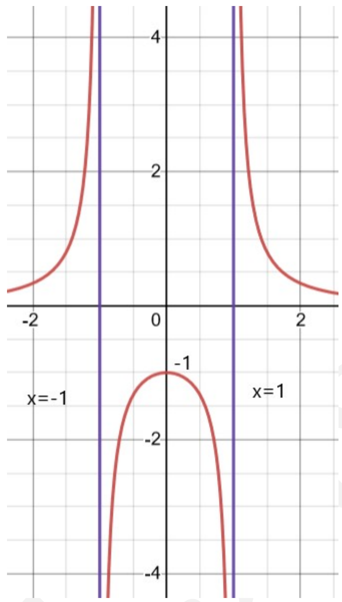
The numbers which make the denominator $= 0$ are $x = \pm 1$, then $D_f = \mathbb{R} \setminus \{\pm 1\}$.

1) y-intersection: $f(0) = -1,\ (0,-1)$

but we can not simplify the function, then we need to change the function $x \to y$, this will be

$$\begin{aligned} y =& \frac{1}{x^2-1}, \\ y(x^2-1) =& 1, \\ yx^2 - y =& 1, \\ x^2 =& \frac{1+y}{y}, \\ x =& \pm\sqrt{\frac{1+y}{y}}. \end{aligned}$$

Which means $D_x = \{y : -1 < y \le 0\}$ makes problem.



$$D_f = \mathbb{R} \setminus \{\pm 1\} \text{ and } R_f = \{y : y \le -1\} \cup \{y : y > 0\}$$

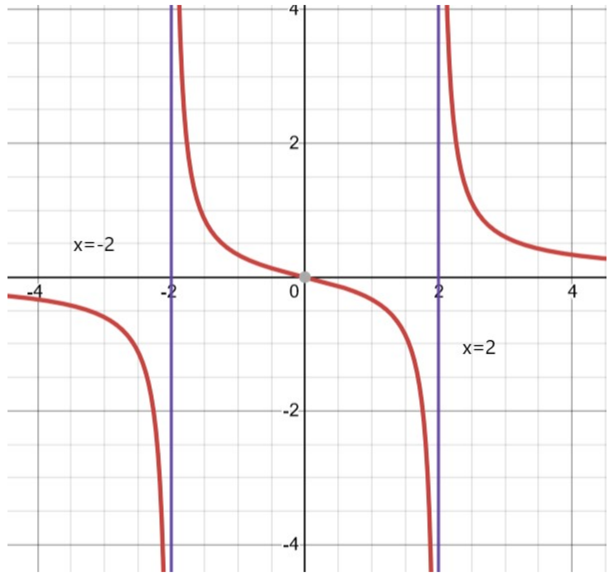
Example 10. Sketch and find the Domain and the Range and find the symmetric point (or line) of the following function:

$$f(x) = \frac{x}{x^2-4}$$

The numbers which make the denominator $= 0$ are $x = \pm 2$, then $D_f = \mathbb{R} \setminus \{\pm 2\}$,

1) y-intersection: $f(0) = -0,\ (0,0)$.

we do not need to change the function $x \to y$



$$D_f = \mathbb{R} \setminus \{\pm 2\} \text{ and } R_f = \mathbb{R}$$

Homework

Question 6. Sketch and find the domain and the range for the following functions:

$$f(x) = \frac{6}{x^2+3}, \qquad f(x) = \frac{x^2-1}{x^2+1}$$

Calculus

التفاضل والتكامل

2024-2025

Lecture 4

Ass. Prof. Dr. Amal Jasim Mohammed

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Limit and Continuity الغاية والاستمرارية

Definition 1. Limits: we write it as

$$\lim_{x \to a} f(x) = L, \qquad L: \text{ constant}$$

which is read تقرأ: the limit of $f(x)$ as x approaches تقترب to a is L

Theorem: Let a and k be numbers,

$$\lim_{x \to a} f(x) = \lim_{x \to a} k = k$$

$$\lim_{x \to a} f(x) = \lim_{x \to a} x = a$$

Theorem: Let a and k be numbers and suppose that

$$\lim_{x \to a} f(x) = L_1, \qquad \lim_{x \to a} g(x) = L_2$$

$$1) \lim_{x \to a} [f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$$
$$= L_1 \pm L_2$$

$$\lim_{x \to 2} [x \pm 3] = \lim_{x \to 2} x \pm \lim_{x \to 2} 3 = 2 \pm 3 = 5 \quad \text{or} \quad -1$$

$$2) \lim_{x \to a} [f(x) \times g(x)] = \lim_{x \to a} f(x) \times \lim_{x \to a} g(x)$$
$$= L_1 \times L_2$$

$$\lim_{x \to 0} [x \times e^x] = \lim_{x \to 0} x \times \lim_{x \to 0} e^x = 0 \times 1 = 0$$

$$3)\ \lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{L_1}{L_2}, \quad L_2 \neq 0$$

$$4)\ \lim_{x\to a}\sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$$
$$= \sqrt[n]{L_1}, \qquad L_1 > 0, \quad \text{when } n \text{ is even}$$

$$\lim_{x\to 1}\sqrt[4]{\frac{1}{x}} = \sqrt[4]{\lim_{x\to 1}\frac{1}{x}} = \sqrt[4]{1} = 1$$

$$5)\ \lim_{x\to a}\ k\times f(x) = k\times \lim_{x\to a} f(x) = k\ L_1$$

$$\lim_{x\to 1} 3\times x = 3\times \lim_{x\to 1} x = 3\times 1 = 3$$

The above statements are true for one-side limit as $x \to a^+$ or $x \to a^-$.

Remark: If n is a positive integer, then

$$\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n = (f(a))^n$$

$$\lim_{x\to -2}(x+1)^2 = \left(\lim_{x\to -2}(x+1)\right)^2 = 1$$

Theorem: For any polynomial $P(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n$ for any real number a

$$\lim_{x\to c} P(x) = a_0c^n + a_1c^{n-1} + \cdots + a_n = P(c).$$

Example 1. Find

$$\lim_{x\to 5} P(x), \qquad \text{where} \qquad P(x) = x^2 - 4x + 3$$

$$\lim_{x\to 5} P(x) = P(5) = 5^2 - 4(5) + 3 = 8$$

Note: A function $f(x)$ has a limit at $x \to a$ if and only if

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L(\text{exists}), \qquad L \quad \text{is constant.}$$

Example 2. Show that $f(x)$ has limit at $x \to 2$

$$\lim_{x\to 2} f(x)? \quad \text{where} \quad f(x) = \begin{cases} \frac{x^2-4}{x-2} & x \geq 2 \\ 4 & x < 2 \end{cases}$$

First,

$$\begin{aligned} \lim_{x\to 2^+} f(x) &= \lim_{x\to 2^+} \frac{x^2-4}{x-2} \\ &= \lim_{x\to 2^+} \frac{(x-2)(x+2)}{x-2} \\ &= \lim_{x\to 2^+} (x+2) = \underline{\underline{4}} \end{aligned}$$

Second,

$$\begin{aligned} \lim_{x\to 2^-} f(x) &= \lim_{x\to 2^-} 4 \\ &= \underline{\underline{4}}. \end{aligned}$$

In this case,

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^-} f(x) = 4$$

Then, $f(x)$ has limit at $x \to 2$

$$\lim_{x\to 2} f(x) = 4$$

Theorem: Consider the rational function $f(x) = \frac{p(x)}{q(x)}$ where $p(x)$ and $q(x)$ are polynomials.

For any real number a:

- if $q(a) \neq 0$,

$$\lim_{x\to a} f(x) = f(a)$$

- if $q(a) = 0$, but $p(a) \neq 0$, then

$$\lim_{x\to a} f(x), \qquad \text{DNE: does not exist.}$$

ليس لها غاية

Example 3. Show that

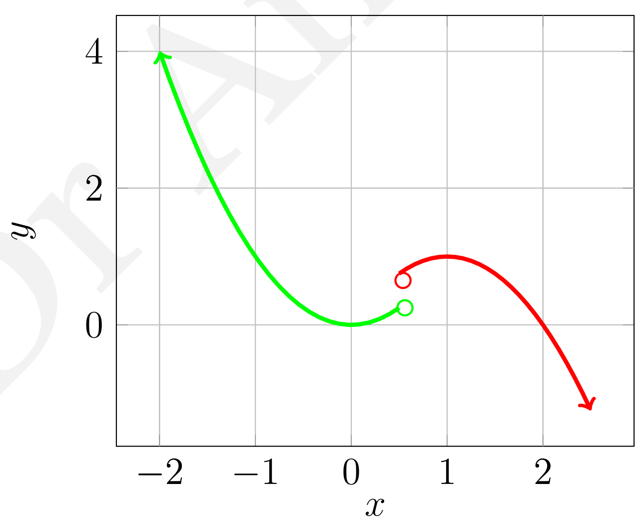
$$\lim_{x\to 3} \frac{x}{\frac{1}{3}x - 1}$$

is not exist. <u>Solution:</u>

$$\lim_{x\to 3} x = 3$$
$$\lim_{x\to 3} \frac{1}{3}x - 1 = 1 - 1 = 0!!$$

Then,

$$\lim_{x\to 3} \frac{x}{\frac{1}{3}x - 1} = ?, \qquad \text{DNE}$$



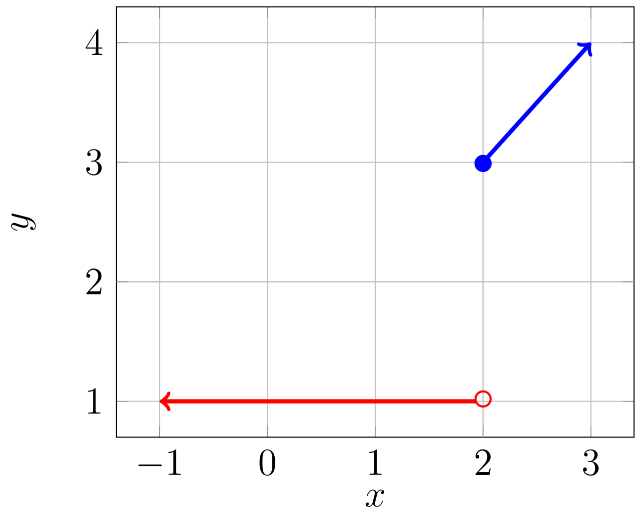
Example 4. For the function f in the picture, the one-side limits

$$\lim_{x\to x_0^+} f(x) \quad \text{and} \quad \lim_{x\to x_0^-} f(x)$$

both are exist <u>but</u> they are not the same. Then

$$\lim_{x\to x_0} f(x)$$

dose not exist (DNE)



Example 5. Check whether $f(x)$ has limit at $x \to 2$

$$\text{where} \quad f(x) = \begin{cases} x+1 & x \geq 2 \\ 1 & x < 2 \end{cases}$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (x+1) = 3$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} 1 = 1$$

In this case,

$$\lim_{x\to 2^+} f(x) \neq \lim_{x\to 2^-} f(x)$$

Then,

$$\lim_{x\to 2} f(x) \quad \text{DNE.}$$

Limit at infinity $\pm\infty$

Remark

$$\lim_{x\to +\infty} x^n = +\infty, \qquad n = 1, 2, 3, 4, \ldots$$

$$\lim_{x\to -\infty} x^n = \begin{cases} -\infty, & n = 1, 3, 5, \ldots \\ +\infty, & n = 2, 4, 6, \ldots \end{cases}$$

$$\lim_{x\to -\infty} (a_0x^n + a_1x^{n-1} + \cdots + a_n) = \lim_{x\to -\infty} a_0x^n$$

$$\lim_{x\to +\infty} (a_0x^n + a_1x^{n-1} + \cdots + a_n) = \lim_{x\to +\infty} a_0x^n$$

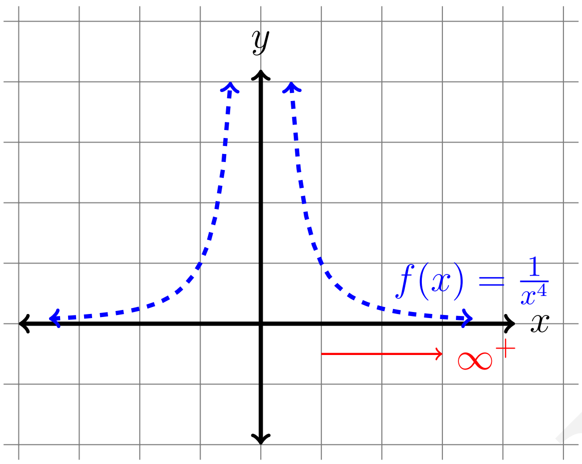
Example 6. Find the limit for the following functions

$$1)\ \lim_{x\to+\infty} -\frac{1}{6}\,x^6 = -(+\infty) = -\infty$$

$$2)\ \lim_{x\to+\infty} 2\,x^2 - \frac{1}{6}\,x^4 = \lim_{x\to+\infty}\left(-\frac{1}{6}\,x^4\right)$$

$$3)\ \lim_{x\to-\infty} 2\,x^5 = -\infty$$

$$4)\ \lim_{x\to-\infty} -\frac{3}{7}\,x^6 = -(+\infty) = -\infty$$

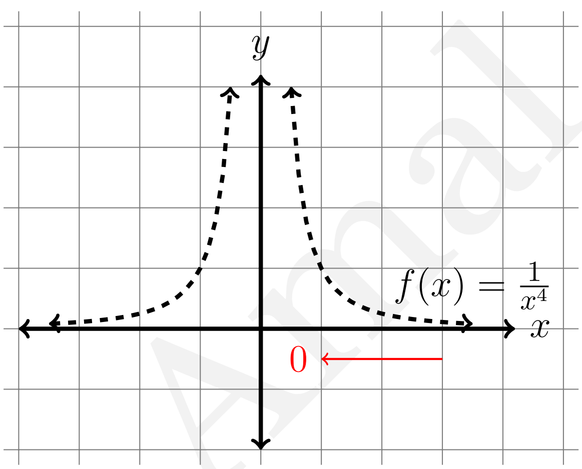


Example 7. Find

$$\lim_{x\to\infty^\pm} f(x), \quad \text{where} \quad f(x) = \frac{1}{x^4}$$

$D_f = \mathbb{R}\backslash\{0\}$ and $R_f = \{y : y > 0\}$

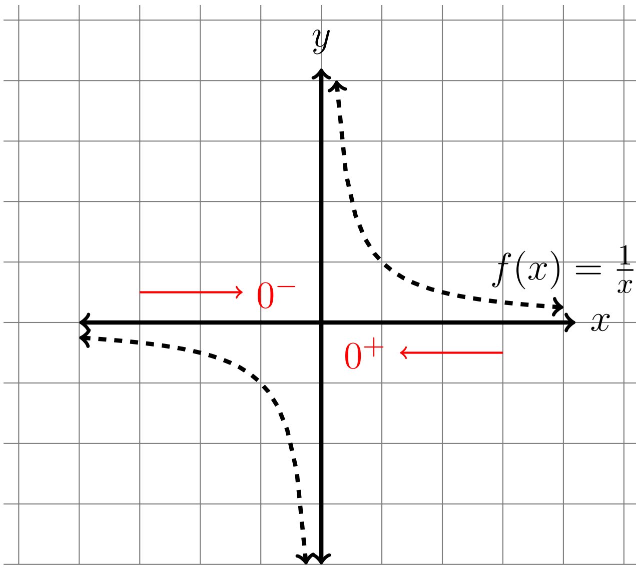
$$\lim_{x\to\infty^+}\frac{1}{x^4} = 0 = \lim_{x\to\infty^-}\frac{1}{x^4}$$



Example 8. Find

$$\lim_{x\to 0^\pm} f(x), \quad \text{where} \quad f(x) = \frac{1}{x^4}$$

$$\lim_{x\to 0^+}\frac{1}{x^4} = \infty^+ = \lim_{x\to 0^-}\frac{1}{x^4}$$

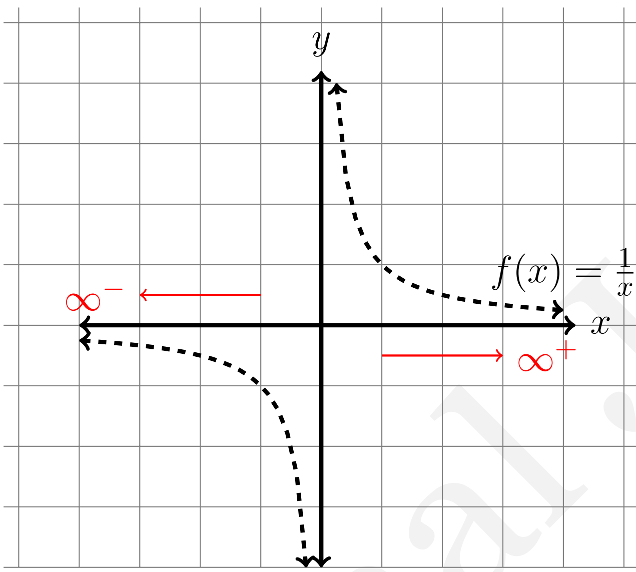


Example 9. Find

$$\lim_{x\to 0^\pm} f(x), \quad \text{where} \quad f(x) = \frac{1}{x}$$

$D_f = \mathbb{R}\backslash\{0\}$ and $R_f = \mathbb{R}\backslash\{0\}$

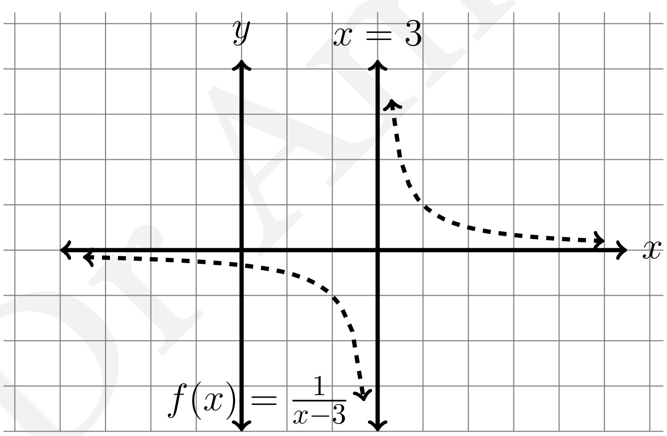
$$\lim_{x\to 0^+} \frac{1}{x} = \infty^+ \qquad \lim_{x\to 0^-} \frac{1}{x} = \infty^-$$



Example 10. Find

$$\lim_{x\to \infty^\pm} f(x), \quad \text{where} \quad f(x) = \frac{1}{x}$$

$$\lim_{x\to\infty^+} \frac{1}{x} = 0 = \lim_{x\to\infty^-} \frac{1}{x}$$

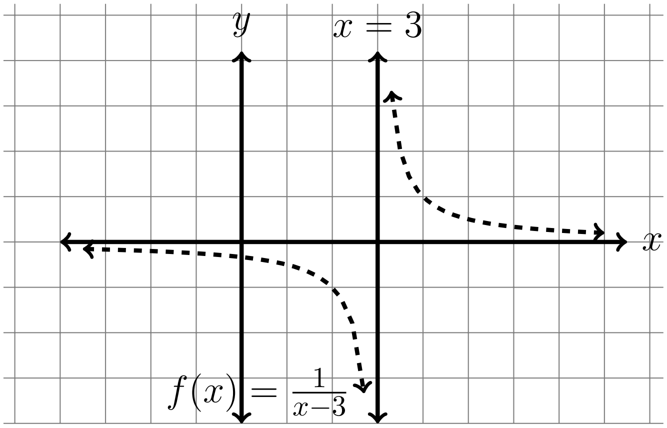


Example 11. Find

$$\lim_{x\to \infty^\pm} f(x), \quad \text{where} \quad f(x) = \frac{1}{x-3}$$

$$D_f = \mathbb{R}\backslash\{3\} \quad \text{and} \quad R_f = \mathbb{R}\backslash\{0\}$$

$$\lim_{x\to\infty^+} \frac{1}{x-3} = 0 = \lim_{x\to\infty^-} \frac{1}{x-3}$$



Example 12. Find

$$\lim_{x \to 3^{\pm}} f(x), \quad \text{where} \quad f(x) = \frac{1}{x-3}$$

$$\lim_{x \to 3^{+}} \frac{1}{x-3} = \infty^{+} \qquad \lim_{x \to 3^{-}} \frac{1}{x-3} = \infty^{-}$$

Homework

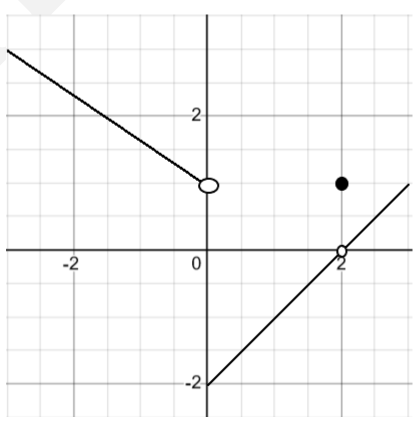
Question 1. Find D_f, R_f, and check whether $f(x)$ has limit for the following functions:

$$\lim_{x \to 3} f(x), \quad \text{where} \quad f(x) = \frac{1}{(x-3)^2}$$

$$\lim_{y \to 6^{-}} f(y), \quad \text{where} \quad f(y) = \frac{y+6}{y^2-36}$$

$$\lim_{y \to 4} f(y), \quad \text{where} \quad f(y) = \frac{4-y}{2-\sqrt{y}}$$

Question 2. Define the function and find the domain and range that represent the following graph



$D_f = ?$ and $R_f = ?$

Trigonometric functions and their limits

Properties of the trigonometric functions :

$$\sin(A \pm B) = \sin(A)\cos(B) \pm \sin(B)\cos(A)$$
$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(B)\sin(A)$$

$$\sin^2(x) + \cos^2(x) = 1$$
$$1 + \cot^2(x) = \csc^2(x)$$
$$\tan^2(x) + 1 = \sec^2(x)$$

$$\sin(2x) = 2\sin(x)\cos(x)$$
$$\cos(2x) = 1 - 2\sin^2(x)$$
$$\cos(2x) = 2\cos^2(x) - 1$$

Theorem: If c is any number in the N domain of the stated trigonometric function, then

$$\lim_{x\to c}\sin(x) = \sin(c) \quad , \quad \lim_{x\to c}\cos(x) = \cos(c), \quad \lim_{x\to c}\tan(x) = \tan(c)$$
$$\lim_{x\to c}\csc(x) = \csc(c) \quad , \quad \lim_{x\to c}\sec(x) = \sec(c), \quad \lim_{x\to c}\cot(x) = \cot(c).$$

Theorem:

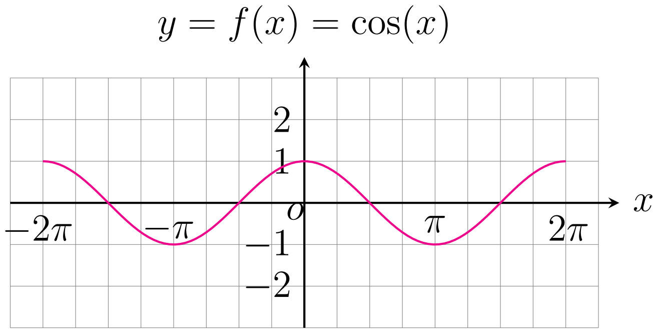
$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1, \qquad \lim_{x\to 0}\frac{1-\cos(x)}{x} = 0$$

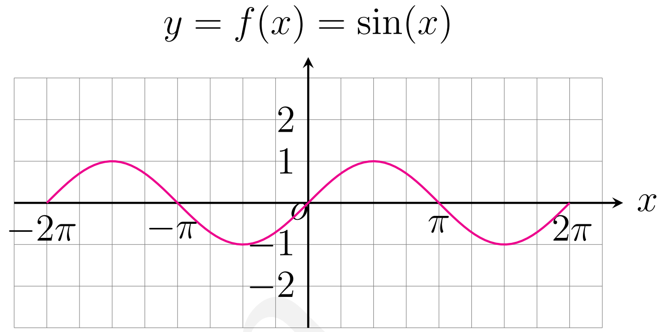
Example 13. Find :

$$\lim_{x\to 0}\frac{\tan(x)}{x}.$$

$$\lim_{x\to 0}\left(\tan(x) \times \frac{1}{x}\right) = \lim_{x\to 0}\left(\frac{\sin(x)}{\cos(x)} \times \frac{1}{x}\right)$$

$$
\begin{aligned}
&= \lim_{x\to 0}\left(\frac{\sin(x)}{x}\times\frac{1}{\cos(x)}\right)\\
&= \lim_{x\to 0}\frac{\sin(x)}{x}\times\lim_{x\to 0}\frac{1}{\cos(x)}\\
&= 1\times 1 = 1
\end{aligned}
$$





Example 14. Find :

$$\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta}.$$

$$
\begin{aligned}
\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta} &= \frac{2}{2}\lim_{\theta\to 0}\frac{\sin(2\theta)}{\theta}\\
&= 2\lim_{\theta\to 0}\frac{\sin(2\theta)}{2\theta}\\
&= 2\times 1 = 2
\end{aligned}
$$

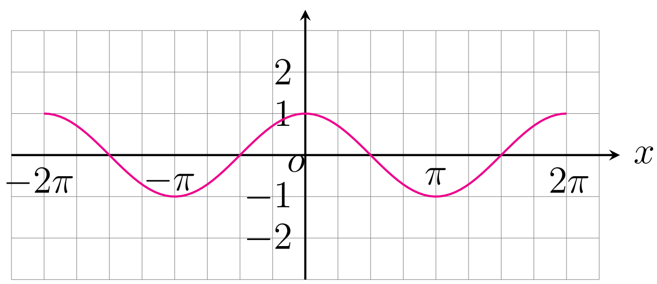
Example 15. Find

$$\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)}$$

$$
\begin{aligned}
\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)} &= \lim_{x\to 0}\frac{\frac{\sin(x)}{\cos(x)}-\sin(x)}{\sin^3(x)}\\
&= \lim_{x\to 0}\left(\frac{\sin(x)-\sin(x)\cos(x)}{\cos(x)}\times\frac{1}{\sin^3(x)}\right)\\
&= \lim_{x\to 0}\left(\frac{\sin(x)(1-\cos(x))}{\cos(x)}\times\frac{1}{\sin^3(x)}\right)
\end{aligned}
$$

$$\begin{aligned}
\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)} &= \lim_{x\to 0}\left(\frac{1-\cos(x)}{\cos(x)}\times\frac{1}{\sin^2(x)}\right), \quad \sin^2(x)+\cos^2(x)=1\\
&= \lim_{x\to 0}\left(\frac{1-\cos(x)}{\cos(x)}\times\frac{1}{1-\cos^2(x)}\right)\\
&= \lim_{x\to 0}\left(\frac{1-\cos(x)}{\cos(x)}\times\frac{1}{(1-\cos(x))(1+\cos(x))}\right)\\
&= \lim_{x\to 0}\left(\frac{1}{\cos(x)}\times\frac{1}{(1+\cos(x))}\right) = 1\times\frac{1}{2}=\frac{1}{2}.
\end{aligned}$$

$y = f(x) = \cos(x)$



Then,

$$\lim_{x\to 0}\frac{\tan(x)-\sin(x)}{\sin^3(x)} = \frac{1}{2}.$$

Calculus

التفاضل والتكامل

2024-2025

Lecture 5

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Find the Domain and Range of the following functions

$$f(x) = [x+1], \qquad D_f = \mathbb{R}, \qquad R_f = \mathbb{Z}$$

$$f(x) = \frac{1}{[x+1]}, \qquad D_f = \mathbb{R} \setminus [-1, 0), \qquad R_f = \left\{ y : y \in \frac{1}{n}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

$$f(x) = \frac{1}{[x]}, \qquad D_f = \mathbb{R} \setminus [0, 1), \qquad R_f = \left\{ y : y \in \frac{1}{n}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

$$f(x) = \frac{1}{[x+0.5]}, \qquad D_f = \mathbb{R} \setminus [-0.5, 0.5), \qquad R_f = \left\{ y : y \in \frac{1}{n}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

$$f(x) = [x-1.2], \qquad D_f = \mathbb{R}, \qquad R_f = \mathbb{Z}$$

$$f(x) = \frac{1}{[x-1.2]}, \qquad D_f = \mathbb{R} \setminus [1.2, 2.2), \qquad R_f = \left\{ y : y \in \frac{1}{n}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

$$f(x) = \frac{1}{[1.2-x]}, \qquad D_f = \mathbb{R} \setminus (0.2, 1.2], \qquad R_f = \left\{ y : y \in \frac{1}{n}, n \in \mathbb{Z} \setminus \{0\} \right\}$$

Applying limit on different types of functions

Example 1.

$$\lim_{x \to 5^-} [x] = \lim_{x \to 5^-} [4.9] = 4$$

Example 2.

$$\lim_{x \to 5^+} [x] = \lim_{x \to 5^+} [5.1] = 5$$

Example 3.

$$\lim_{x \to (-4.2)^+} [x] = \lim_{x \to (-4.2)^+} [-4.1] = -5$$

Example 4.

$$\lim_{x \to (-4.2)^-} [x] = \lim_{x \to (-4.2)^-} [-4.3] = -5$$

Example 5.

$$\lim_{x\to(4.2)^+}[x] = \lim_{x\to(4.2)^+}[4.3] = 4$$

Example 6.

$$\lim_{x\to(4.2)^-}[x] = \lim_{x\to(4.2)^-}[4.1] = 4$$

Homework

Question 1. Find the D_f and R_f and the limit when s approaches 1.

$$f(s) = \frac{1}{[s-1]}$$

Example 7.

$$\lim_{x\to0^-}\frac{|x|}{x} = \lim_{x\to0^-}\frac{-x}{x} = \lim_{x\to0^-} -1 = -1$$

Example 8.

$$\lim_{x\to0^+}\frac{|x|}{x} = \lim_{x\to0^+}\frac{x}{x} = \lim_{x\to0^+} 1 = 1$$

Example 9.

$$\lim_{x\to1^+}\frac{x^4-1}{x-1} = \lim_{x\to1^+}\frac{(x^2-1)(x^2+1)}{x-1} = \lim_{x\to1^+}\frac{(x-1)(x+1)(x^2+1)}{x-1} = 4$$

Question 2. Let

$$f(x) = \begin{cases} \frac{1}{x+2} & x < -2 \\ x^2-5 & -2 \le x \le 3 \\ \sqrt{x+13} & x > 3 \end{cases}$$

Check and find if it is possible

1 $\lim_{x\to-2} f(x)$.

2 $\lim_{x\to0} f(x)$.

3 $\lim_{x\to3} f(x)$.

Solution 1.

Sol.1

بماانه رقم ٢ معرفةعلى دالتين مختلفتين وباتجاهين مختلفين يجب ان نختبرالجهتين

$$\lim_{x\to -2^-} f(x) = \lim_{x\to -2^-} \frac{1}{x+2} = -\infty$$
$$\lim_{x\to -2^+} f(x) = \lim_{x\to -2^+} x^2 - 5 = -1$$

Thus, the $\lim_{x\to -2} f(x)$ DNE.

Sol.2

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 - 5 = -5$$

Sol.3

بماانه رقم ٣ معرفةعلى دالتين مختلفتين وباتجاهين مختلفين يجب ان نختبرالجهتين

$$\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x^2 - 5 = 4$$
$$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} \sqrt{x+13} = 4$$

Thus, the $\lim_{x\to 3} f(x)$ is exist =4.

If we want to find the *lim* of a rational function $f(x)$ at $x \to \infty$, and the substitution is $\frac{\infty}{\infty}$, we can divide the numerator and the denominator by the highest indices (power) of x.
اذا لدينا دالة كسرية وفي حالة التعويض لل ∞ ووجدنا بان البسط والمقام هو $\frac{\infty}{\infty}$ نقسم البسط والمقام على اكبر اس لـــ x بالمقام.

Example 10.

$$\lim_{x\to+\infty} \sqrt[3]{\frac{3x+5}{6x-8}} = \lim_{x\to+\infty} \sqrt[3]{\frac{3+\frac{5}{x}}{6-\frac{8}{x}}} = \sqrt[3]{\lim_{x\to+\infty} \frac{3+\frac{5}{x}}{6-\frac{8}{x}}} = \sqrt[3]{\frac{1}{2}} = \frac{1}{\sqrt[3]{2}}$$

Homework

Question 3. Find and check whether $f(x)$ has limit for the following functions:

$$\lim_{x\to+\infty} f(x), \quad \text{where} \quad f(x)=\frac{\sqrt{x^2+2}}{3x-6}$$

$$\lim_{x\to-\infty} f(x), \quad \text{where} \quad f(x)=\frac{\sqrt{x^2+2}}{3x-6}$$

Continuity

Definition 1. *A function $f(x)$ is said to be continuous at $x=c$, if the following conditions are hold* يجب ان تتحقق الشروط الثلاث ادناه*:*

1. *$f(x)$ is defined at $x=c$.*
2. *The limit of $f(x)$ at $x\to c$ must be exist, which means*

$$\lim_{x\to c^+} f(x)=\lim_{x\to c^-} f(x).$$

3. *The limit of $f(x)$ at $x\to c$ must be equal to $f(c)$, which means*

$$\lim_{x\to c} f(x)=f(c).$$

If one or more condition of the above definition fails يفشل to hold يتحقق then, we will say that $f(x)$ has a discontinuity غير مستمرة at $x=c$.

Example 11. Prove (show) that $f(x)=x^2+x+10$ is continuous at $x=2$.
نحتاج ان نطبق الشروط الثلاث اعلاه

1. $f(x)$ at $x=2$, $f(2)=(2)^2+2+10=16$.
2. لكون الدالة متعددة حدود: الغاية من اليمين والغاية من اليسار معرفة

$$\lim_{x\to 2^\pm} f(x)=(2)^2+2+10=16.$$

3.

$$\lim_{x\to 2} f(x) = f(2) = 16.$$

Then, $f(x)$ is continuous at $x = 2$.

Example 12. Show that

$$f(x) = \begin{cases} \frac{x^2-9}{x-3}, & x \neq 3 \to x \gtrless 3 \\ 8, & x = 3 \end{cases}$$

has a discontinuity at $x = 3$.

1. $f(3) = 8$.
2. We need to find the limit of $f(x)$ at $x \to 3$:

$$\begin{aligned} \lim_{x\to 3^{\pm}} f(x) &= \lim_{x\to 3^{\pm}} \frac{x^2-9}{x-3} \\ &= \lim_{x\to 3^{\pm}} \frac{(x-3)(x+3)}{x-3} \\ &= \lim_{x\to 3^{\pm}} (x+3) = 6 \end{aligned}$$

3.

$$\lim_{x\to 3} f(x) \neq f(3).$$

Then, $f(x)$ has a discontinuity at $x = 3$.

Example 13. Show that

$$f(x) = \begin{cases} \frac{x^3-1}{x-1}, & x \neq 1 \to x \gtrless 1 \\ 3, & x = 1 \end{cases}$$

is continuous at $x = 1$.

1. $f(1) = 3$

2. We need to find the limit of $f(x)$ at $x \to 1$:

$$\lim_{x\to 1^\pm} f(x) = \lim_{x\to 1^\pm} \frac{x^3-1}{x-1} = \lim_{x\to 1^\pm} \frac{(x-1)(x^2+x+1)}{x-1} = \lim_{x\to 1^\pm}(x^2+x+1) = 3$$

3.

$$\lim_{x\to 1} f(x) = f(1) = 3.$$

Then, $f(x)$ is continuous at $x = 1$.

Example 14. Check whether this function is continuous or has a discontinuity at $x = 0$, where

$$f(x) = \begin{cases} \frac{x}{\sqrt{x+1}-1}, & x > 0 \\ x+2, & x \le 0 \end{cases}$$

1. $f(0) = 0 + 2 = 2$

2. We need to find the limit of $f(x)$ at $x \to 0$:

- the limit of $f(x)$ at $x \to 0^+$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{\sqrt{x+1}-1} = \lim_{x\to 0^+} \frac{x}{\underline{\underline{\sqrt{x+1}-1}}} \times \frac{\sqrt{x+1}+1}{\underline{\underline{\sqrt{x+1}+1}}} = \lim_{x\to 0^+} \frac{x(\sqrt{x+1}+1)}{x+\underline{1}-\underline{1}} = \lim_{x\to 0^+} \frac{x(\sqrt{x+1}+1)}{x} = \lim_{x\to 0^+} \sqrt{x+1}+1 = 2$$

- the limit of $f(x)$ at $x \to 0^-$

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x+2) = 2$$

3.

$$\lim_{x\to 0} f(x) = f(0) = 2.$$

Then, $f(x)$ is continuous at $x = 0$.

Example 15. Is $f(x) = [x + 0.5]$ continuous at $x = 0$?

1. $f(0) = [0 + 0.5] = [0.5] = 0$.

2. We need to find the limit of $f(x)$ at $x \to 0$:

   - the limit of $f(x)$ at $x \to 0^+$

$$\begin{aligned}\lim_{x\to 0^+} f(x) &= \lim_{x\to 0^+} [x + 0.5] \\ &= \lim_{x\to 0^+} [0 + 0.5] = [0.51] = 0\end{aligned}$$

   - the limit of $f(x)$ at $x \to 0^-$

$$\begin{aligned}\lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} [x + 0.5] \\ &= \lim_{x\to 0^-} [0 + 0.5] = [0.49] = 0\end{aligned}$$

3.

$$\lim_{x\to 0} f(x) = f(0) = 0.$$

Then, $f(x)$ is continuous at $x = 0$.

Homework

Question 4. *Is $f(x)$ continuous at $x = 9$?*

$$f(x) = \begin{cases} \frac{x-9}{\sqrt{x}\;-3}, & x > 9 \\ 6, & x \le 9 \end{cases}$$

Example 16. Find the value of k which makes the function $f(x)$ (if it is possible) continuous at $x = 1$, where k is a constant and

$$f(x) = \begin{cases} 7x - 2, & x \leq 1 \\ k\ x^2, & x > 1 \end{cases}$$

Since $f(x)$ is continuous at $x = 1$, then the conditions must be hold which are

1. $f(1) = (7 \times 1 - 2) = 5.$

2. The limit of $f(x)$ is exist at $x \to 1$ which means

$$\begin{aligned} \lim_{x \to 1^+} f(x) &= \lim_{x \to 1^-} f(x) \\ \lim_{x \to 1^+} k\ x^2 &= \lim_{x \to 1^-} (7x - 2) \\ k &= 5. \end{aligned}$$

3. the limit of $f(x)$ when $x \to 1$ equal to $f(1)$ when $k = 5$.

Then, the function $f(x)$ is continuous at $x = 1$ when $k = 5$.

Example 17. Find the value of k which makes the function $f(x)$ (if it is possible) continuous at $x = 2$, where k is a constant and

$$f(x) = \begin{cases} k\ x^2, & x \leq 2 \\ 2x + k, & x > 2 \end{cases}$$

Since $f(x)$ is continuous at $x = 2$, then the conditions must be hold which are

1. The limit of $f(x)$ is exist at $x \to 2$ which means

$$\begin{aligned} \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^-} f(x) \\ \lim_{x \to 2^+} (2x + k) &= \lim_{x \to 2^-} (k\ x^2) \\ 4 + k &= 4k \\ 4 &= 4k - k \\ k &= \frac{4}{3}. \end{aligned}$$

2. $f(2) = k\ x^2 = \frac{4}{3} \times 4 = \frac{16}{3}$.

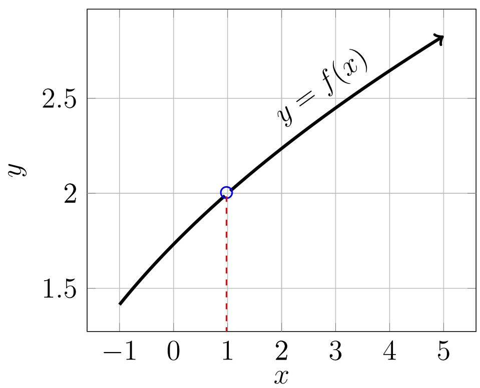
3.

$$\lim_{x\to 2^+} (2x + k) = \lim_{x\to 2^+} 2x + \frac{4}{3} = \frac{16}{3}.$$

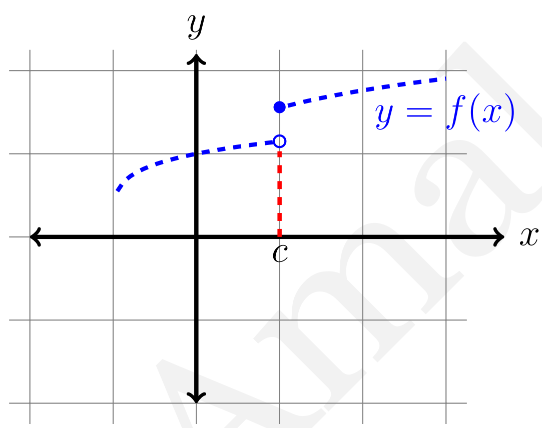
$$\lim_{x\to 2^-} (k\ x^2) = \lim_{x\to 2^-} \frac{4}{3} \times x^2 = \frac{16}{3}.$$

Then, the function $f(x)$ is continuous at $x = 2$ when $k = \frac{4}{3}$.

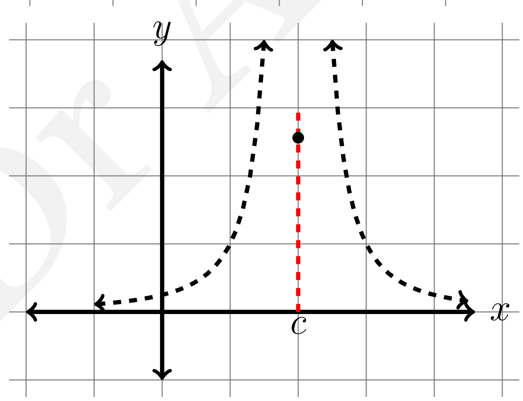
The following Figures illustrate a discontinuity at $x = c$:



The function is not defined at $x = 1$, then the first condition of the defenition does not satisfy. Then, the function has a discontinuity at $x = 1$.



The function is defined at $x = c$, but the limit of $f(x)$ when $x \to c$ DNE. Then the function has a discontinuity at $x = c$.



The function is defined at $x = c$, but the limit of $f(x)$ when $x \to c$ DNE. Then the function has a discontinuity at $x = c$.

Example 18. Define $g(4)$ in away that extends توسيع

$$g(x) = \frac{x^2 - 16}{x^2 - 3x - 4}$$

to be continuous at $x = 4$.

Since we need $g(x)$ to be continuous at $x = 4$, first we need to simplify $g(x)$

$$\begin{aligned} g(x) =& \frac{x^2 - 16}{x^2 - 3x - 4} \\ =& \frac{(x-4)(x+4)}{(x-4)(x+1)} \\ =& \frac{x+4}{x+1} \end{aligned}$$

Then

$$\lim_{x \to 4^{\pm}} g(x) = \frac{8}{5}, \quad \text{which means} \quad \lim_{x \to 4} g(x) \quad \text{is exist.}$$

So, we can define

$$g(x) = \begin{cases} \frac{x^2-16}{x^2-3x-4} & x \neq 4 \\ \frac{8}{5} & x = 4 \end{cases}$$

to be continuous at $x = 4$.

Example 19. Make $h(x)$ continuous at $x = 2$ by extending the function $h(x)$. Where

$$h(x) = 7 - 2x.$$

First we need to find $h(2) : h(2) = 7 - 2(2) = 3$. In this case we can define the function $h(x)$ as

$$h(x) = \begin{cases} 7 - 2x & x \neq 2 \\ 3 & x = 2 \end{cases}$$

to be continuous at $x = 2$.