



جامعة الموصل  
كلية التربية للعلوم الصرفة  
قسم الفيزياء



Advanced Mathematics

رياضيات متقدمة

المرحلة الثانية  
مدرس المادة: م.م. محمد علي محمد الوزان

المحاضرة الأولى

المسلسلات

The Series

## **Arithmetic Series:**

The sum of the terms of an arithmetic sequence is called as arithmetic series.

$7, 17, 27, 37, 47, \dots \Rightarrow$  is an Arithmetic sequence

$7 + 17 + 27 + 37 + 47 + \dots \Rightarrow$  is Arithmetic series

## **The sum of $n$ terms of an Arithmetic Series:**

$$S_n = n \left[ \frac{a_1 + a_n}{2} \right], a_n = a_1 + (n - 1)d$$

**Where:**

$a_1$ : is the first term

$a_n$ : is the  $n^{\text{th}}$  term

$n$ : is the number of terms

$d$ : is the common difference

**Example 1:** Find the sum of the sequence 3, 9, 15, 21, ... to 10 terms.

**Solution:**

$$S_n = \frac{n}{2}[2a_1 + (n-1)d], \quad a_1 = 3$$

$$d = a_n - a_{n-1} = 9 - 3 = 6$$

$$S_{10} = \frac{10}{2}[2(3) + (10-1)6] \Rightarrow S_{10} = 5[6 + 54]$$

$$S_{10} = 5[60] = 300$$

**Example 2:** Find the sum of the 60 terms of the following series

$$9 + 14 + 19 + 24 + \dots + 289 + 294 + 299 + 304$$

**Solution:**

$$d = a_n - a_{n-1} = 24 - 19 = 5, \quad n = 60, \quad a_1 = 9, \quad a_{60} = 304$$

$$1. \quad S_n = \frac{n}{2}[2a_1 + (n-1)d] \Rightarrow S_{60} = \frac{60}{2}[2(9) + (60-1)(5)]$$

$$S_{60} = 30[18 + 59(5)] \Rightarrow S_{60} = 30(313) = 9390$$

$$2. \quad S_n = n \left[ \frac{a_1 + a_n}{2} \right] \Rightarrow S_{60} = 60 \left[ \frac{9 + 304}{2} \right] \Rightarrow S_{60} = 30(313)$$

$$S_{60} = 9390$$

## Geometric Series:

The geometric series is a series that has the form  $\sum_{n=0}^{\infty} ar^n$ , where  $(a)$  is a real constant and  $(r)$  is a real number.

$$\sum_{n=0}^{\infty} ar^n = S_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$$

**Example:** find the sum of the series  $\sum_{k=1}^{\infty} \frac{1}{2^k}$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

**Example:** Write out the first four terms of the series

$$\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n$$

$$\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n = \underbrace{\frac{8}{27}}_a + \underbrace{\frac{16}{81}}_{ar} + \underbrace{\frac{32}{243}}_{ar^2} + \underbrace{\frac{64}{729}}_{ar^3} + \cdots$$

المحاضرة الثانية  
الغابات

The Limit

## Rules of limits: Useful rules for finding limits

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}, \text{ where } g(x) \neq 0$$

### Example:

$$\lim_{x \rightarrow 1} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow 1} (x^4 + x^2 - 1)}{\lim_{x \rightarrow 1} (x^2 + 5)}$$

$$\frac{\lim_{x \rightarrow 1} (x^4) + \lim_{x \rightarrow 1} (x^2) - \lim_{x \rightarrow 1} (1)}{\lim_{x \rightarrow 1} (x^2) + \lim_{x \rightarrow 1} (5)} + \frac{1^4 + 1^2 - 1}{1^2 + 5} = \frac{1}{6}$$

### • Rule no.1

1. If the degree of the numerators is lower than the degree of the denominator, the numerator be expressed as  $n(x)$ , and the denominator be expressed as  $d(x)$ , then:

$$\lim_{x \rightarrow \infty} \frac{n(x)}{d(x)} = 0$$

**Example:**

$$\lim_{x \rightarrow \infty} \frac{5x^3 - 2x^2 + 17x - 13}{12 - 2x + x^4} = \lim_{x \rightarrow \infty} \frac{5x^3}{x^4} = \lim_{x \rightarrow \infty} \frac{5}{x} = 0$$

**Note:** Any value divided by infinity equals 0

2. If the degree of the numerator is higher than the degree of denominator, then:

$$\lim_{x \rightarrow \infty} \frac{n(x)}{d(x)} = \infty \text{ or } -\infty$$

**Example:**

$$\lim_{x \rightarrow \infty} \frac{3x^5 + x^2 - 5}{6 + x + 7x^2} = \lim_{x \rightarrow \infty} \frac{3x^5}{7x^2} = \lim_{x \rightarrow \infty} \frac{3}{7}x^3 = \infty$$

3. If the degree of the numerator is equal the degree of denominator, then is equal to the leading coefficient of  $n(x)$  over the leading coefficient of  $d(x)$ .

**Example:**

$$\lim_{x \rightarrow \infty} \frac{3x^3 - 2x^2 + x - 4}{4 + 2x - 5x^3} = \lim_{x \rightarrow \infty} \frac{3x^3}{-5x^3} = \lim_{x \rightarrow \infty} -\frac{3}{5} = -\frac{3}{5}$$

- **Rule no.2**

**L'Hôpital's Rule:** It is a rule used in Limits, when the results of the Limits are not specified

If the  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  results in one of the following forms:

$0, \pm \infty, \infty \pm \infty$ , And  $\lim_{x \rightarrow c} \frac{f'(x)}{g'(x)}$  exists and  $g'(x) \neq 0$ , then:

$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow c} f'(x)}{\lim_{x \rightarrow c} g'(x)} \text{ This is called L'Hôpital's Rule}$$

**Example 1:** Find the  $\lim_{x \rightarrow 0} \frac{e^x - 1}{x}$

**Solution:**

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = \frac{\lim_{x \rightarrow 0} (e^x - 1)}{\lim_{x \rightarrow 0} (x)} = \frac{0}{0}$$

Using L'Hôpital's Rule

$$\lim_{x \rightarrow 0} \frac{e^x}{1} \Rightarrow \lim_{x \rightarrow 0} \frac{1}{1} = 1$$

**Example 2:** Find  $\lim_{x \rightarrow 0} \frac{e^x - 1 - x - (x^2/2)}{x^3}$

**Solution:**

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x - (x^2/2)}{x^3} = \frac{0}{0}, \text{ Using L'Hôpital's Rule}$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{3x^2} = \frac{0}{0}, \text{ Using L'Hôpital's Rule again}$$

$$\lim_{x \rightarrow 0} \frac{e^x - 1}{6x} = \frac{0}{0}, \text{ Using L'Hôpital's Rule} \quad \lim_{x \rightarrow 0} \frac{e^x}{6} = \frac{e^0}{6} = \frac{1}{6}$$

المحاضرة الثالثة  
تقارب وتباعد المتسلاط  
Convergence and Divergence of Series

## 1. Convergence and Divergence of Arithmetic Series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$S_n = \frac{n}{2} [a_1 + a_n] \text{ or } S_n = \frac{n}{2} [2a_1 + (n-1)d]$$

$$S_n = \begin{cases} \text{Convergent, where } \lim_{n \rightarrow \infty} S_n = L \\ \text{Divergent, where } \lim_{n \rightarrow \infty} S_n = \pm\infty \end{cases}$$

$a_1$ : The first term.

$S_n$ : The sum of terms from 1 to  $n$ .

$L$ : Specific number.

**Example 1:** Test the following series for convergence or divergence  $\sum_{n=1}^{\infty} (2n - 1)$

**Solution:**

$$\sum_{n=1}^{\infty} (2n - 1) = 1 + 3 + 5 + 7 + \cdots + (2n - 1) + \cdots$$

$$d = a_n - a_{n-1} = 5 - 3 = 7 - 5 = 2 = d, \text{ and } a_1 = 1$$

$$S_n = \frac{n}{2} [a_1 + a_n] \Rightarrow S_n = \frac{n}{2} [1 + (2n - 1)] \Rightarrow S_n = \frac{n}{2} [1 + 2n - 1]$$

$$S_n = \frac{n}{2} [2n] \Rightarrow S_n = n^2 \Rightarrow \lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} n^2 = \infty$$

$\sum_{n=1}^{\infty} (2n - 1)$  is divergent.

## 2. Convergence and Divergence of Geometric Series:

$$\sum_{n=1}^{\infty} ar^n = ar + ar^2 + ar^3 + \cdots + ar^n$$

$$r = \begin{cases} |r| < 1, \text{ then the series is convergent, and } S_n = \frac{a}{1-r} \\ |r| \geq 1, \text{ then the series is divergent, and It has no sum} \end{cases}$$

*a: the first term.*

*r: The result of dividing each term by the previous term.*

*S<sub>n</sub>: The sum of terms from 1 to n.*

**Example 1:** Test the following series for convergence or divergence  $\sum_{i=1}^{\infty} \frac{1}{2^i}$ ,

*if convergent then find the sum of this series.*

**Solution:**

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} + \cdots$$

$$r = \frac{a_n}{a_{n-1}} = \frac{\frac{1}{2^2}}{\frac{1}{2^1}} = \frac{\frac{1}{2^3}}{\frac{1}{2^2}} = \frac{1}{2}, |r| = \frac{1}{2} < 1, \text{ then it is convergent.}$$

$$S_n = \frac{a}{1-r} \Rightarrow S_n = \frac{\frac{1}{2}}{1 - \frac{1}{2}} \Rightarrow S_n = \frac{\frac{1}{2}}{\frac{1}{2}} \Rightarrow S_n = \frac{1}{2} * 2 = 1$$

Therefore the series  $\sum_{i=1}^{\infty} \frac{1}{2^i}$  is convergent, and the sum is 1.

### 3. Convergence and Divergence of P-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, p: \text{is constant number.}$$

$$p = \begin{cases} p > 1, \text{then the series is convergent} \\ 0 < p \leq 1, \text{then the series is divergent} \end{cases}$$

**Example 1:** Show that  $\sum_{r=1}^{\infty} \frac{1}{r^2}$  is convergent?

#### Solution:

This series is  $p$  – series, where  $p = 2 > 1$

therefore the series  $\sum_{r=1}^{\infty} \frac{1}{r^2}$  is convergent.

المحاضرة الرابعة  
متسلسلة القوى ومتسلسلة تايلر

Power series and Taylor series

## Centered of Power Series:

This series  $\sum_{n=0}^{\infty} a_n(x - c)^n$  is centered at  $(c)$ , where  $c = a = x$

Example: where the following series are centered?

1.  $\sum_{n=0}^{\infty} x^n$  **centered at  $c = 0$**

2.  $\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n (x - 3)^n$  **centered at  $c = 3$**

## Radius and Interval of convergence of the power series:

for any power series there are only 3 possibilities for the value of  $(x)$  for which the series converges:

1.  $|x| * \infty = \begin{cases} \text{convergent only for } x = 0 \\ \text{Interval} = 0 \\ \text{Radius} = 0 \end{cases}$

2.  $|x| * 0 = \begin{cases} \text{convergent for all value of } x \\ \text{interval} = (-\infty, +\infty) \\ \text{Radius} = \infty \end{cases}$

3.  $|x| * \text{number}$

**Example 5:** Determine the radius and interval of convergence for  $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$

**Solution:**

**Using Ratio Test:**  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} \Rightarrow \lim_{n \rightarrow \infty} \left| \frac{(x+1)^{n+1}}{(n+1)!} * \frac{n!}{(x+1)^n} \right|$

$$= \lim_{n \rightarrow \infty} \left| \frac{x+1}{n+1} \right| = |x+1| \lim_{n \rightarrow \infty} \frac{1}{n+1} = |x+1| * 0 = 0 < 1$$

The series converges for all value of  $x$ .

Interval =  $(-\infty, \infty)$



Radius =  $\infty$

**Example 6:** Find the radius and interval of convergence of the series  $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$

**Solution:**

**Using Ratio Test:**  $\lim_{n \rightarrow \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \rightarrow \infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)^2}}{\frac{(x-2)^n}{n^2}} \right| = \lim_{n \rightarrow \infty} \frac{(x-2)^{n+1}}{(n+1)^2} * \frac{n^2}{(x-2)^n}$

$$= |x-2| \lim_{n \rightarrow \infty} \left( \frac{n}{n+1} \right)^2 = |x-2| * 1 = |x-2|$$

The series converges when  $|x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3$

The Interval =  $(1,3)$



## Taylor Series

A Taylor series is an infinite sum that represents a particular function. The Taylor series for a function  $f(x)$ , centered at  $x = a$ , is the infinite series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$

$$f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n$$

**Note:**  $0! = 1$ .

**Example 1:** Find the first five terms of the Taylor series for:  $f(x) = \sin x$ , centered at  $a = 0$ ?

**Solution:**

$$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$$

$$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$$

$$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$$

$$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$$

$$f''''(x) = \sin x \Rightarrow f''''(0) = \sin 0 = 0$$

$$\begin{aligned} & f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \cdots + \frac{f^{(n)}(a)}{n!} (x - a)^n \\ &= f(0) + f^{(1)}(0)(x - 0)^1 + \frac{f^{(2)}(0)}{2} (x - 0)^2 + \frac{f^{(3)}(0)}{3!} (x - 0)^3 + \cdots \end{aligned}$$

The Taylor series for  $f(x) = \sin x$ , centered at  $(a = 0)$  is:

$$0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + 0 + \frac{x^9}{9!} - \cdots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

المحاضرة الخامسة  
المعادلات التفاضلية الجزئية  
Partial Differential Equation

- **First Order Partial Derivatives:**

$\frac{\partial z}{\partial x}$  is read as partial derivative of z with respect to x

and means differentiate with respect to x holding y constant.

$\frac{\partial z}{\partial y}$  is read as partial derivative of z with respect to y

and means differentiate with respect to y holding x constant.

Another common notation is the subscript notation:

$Z_x$  means  $\frac{\partial z}{\partial x}$  and  $Z_y$  means  $\frac{\partial z}{\partial y}$

**Example 1:** Calculate  $\frac{\partial z}{\partial x}$  and  $\frac{\partial z}{\partial y}$  when  $z = x^2 + 3xy + y - 1$

**Solution:**

$$\frac{\partial z}{\partial x} = 2x + 3y$$

$$\frac{\partial z}{\partial y} = 3x + 1$$

- **Second Order Partial Derivatives:**

We express the second partial derivative in the form:

$$\frac{\partial^2 z}{\partial x^2} = z_{xx}, \quad \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = z_{xy} = \frac{\partial}{\partial y} \left( \frac{\partial z}{\partial x} \right), \quad \frac{\partial^2 z}{\partial y \partial x} = z_{yx} = \frac{\partial}{\partial x} \left( \frac{\partial z}{\partial y} \right)$$

$\frac{\partial^2 z}{\partial x^2}$  means the second derivative with respect to x holding y constant.

$\frac{\partial^2 z}{\partial y^2}$  means the second derivative with respect to y holding x constant.

$\frac{\partial^2 z}{\partial x \partial y}$  means differentiate first with respect to x and then with respect to y.

$\frac{\partial^2 z}{\partial y \partial x}$  means differentiate first with respect to y and then with respect to x.

**Example 1:** Find all second order partial derivatives of the following

function  $f(x, y) = x^2y^2 + y^2 + 2yx^3$

**Solution:**

$$\frac{\partial f}{\partial x} = 2xy^2 + 6x^2y$$

$$\frac{\partial f}{\partial y} = 2x^2y + 2y + 2x^3$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^2 + 12xy$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^2 + 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 4xy + 6x^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 4xy + 6x^2$$