Ministry of Higher Education and Scientific
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Classic Mechanics

الميكانيك الكلاسيكي

The First Stage

المرحلة الاولى

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The First Lecture

Physical Quantity, Units and Vectors

Physical Quantity

Physics is an empirical study. Everything we know that the physical world and about the principles that govern its behavior has been learned through observations of the phenomena of nature. The ultimate test of any physical theory is its agreement with observations and measurements of physical phenomena. Thus physics is inherently a science of measurement.

Any number or set of numbers used for a quantitative description of a physical phenomenon is called a **physical quantity**. To define a physical quantity we must either specify a procedure for measuring the quantity from other quantities that can be measured.

There are two types of physical quantities:-

1- Scalar Quantity:- is one that has nothing to do with spatial direction. Many physical concepts such as length, time, temperature, mass, density, charge, and volume are ***scalars***. Each has a scale or size, but no associated direction. **For examples**, the number of students in a class, the quantity of sugar in a jar, and the cost of a house are familiar scalar quantities.

2- Vector Quantity:- is one that can be specified completely only if we provide both its magnitude and direction. Many physical concepts such as displacement, velocity, acceleration, force, and momentum are ***vector quantities***.

For example, a vector displacement might be a change in position from a certain point to a second point 2 cm away and in the x-direction from the first point.

Units

The quantitative measure of a physical quantity is a number which expresses the ratio of the magnitude of the quantity to the magnitude of an arbitrarily chosen standard amount of the same kind is called **unit of the physical quantity**.

A complete description of the a physical quantity therefore, requires:-

1) The choice of a unit in which the quantity is to be measured.
2) A number which states how many times, this unit, the quantity in questions contains.
3) The measure number, as it is called, depends upon the size of the unit chosen.

For example, the duration of a day is 24 when expressed in hours. It is (24×60) when expressed in minutes, it is (1/365) when expressed in years. Therefore:-

$$n \propto \frac{1}{u} \quad \Longrightarrow \quad nu = Constant = Q$$

Or $$n_1 u_1 = n_2 u_2$$

Where, (n) measure number; (u) units; (Q) physical quantity. Since, physical quantities are related each to other, therefore, they selected the units a limited number, and we can find the units of the other quantities

The few quantities selected for this purpose are called **the basic or fundamental units**. The units of the other quantities can be derived, and hence are called **derived units**.

The basic units listed in table.

N	Quantity	Unit	Symbol
1	Length	meter	m
2	Mass	Kilogram	kg
3	Time	Second	s
4	Electric Current	Ampere	A
5	Temperature	Kelvin	K
6	Amount of substance	Mole	mol

Some derived units usually uses in mechanics.

N	Quantity	Unit	Symbol
1	Force	Newton	N
2	Energy	Joule	J
3	Power	Watt	W
4	Velocity	v	m/sec
5	Pressure	Newton/area	N/m^2
6	Torque	Newton-meter	N.m

Systems of Units

The basic systems are:-

1. The British system

Where:- (F) **Foot**; unit of **length**; (P) **Pound**; unit of mass; **Second** (s); unit of time.

2- The metric system

Where:- (C) **Centimeter**; unit of **length** (cm); (G) **Gram** (gm); unit of **mass**; **Second** (s); unit of **time**.

3- In 1960, the eleventh general conference of weights and measures, on units, proposed revised metric system called the **System International of Unit** in French (**abbreviated**, **SI**) which uses the **meter (m) for length**, the **kilogram (kg) for mass**, and the **second (sec) for time**.

- **Length**:- (meter; m) The length equal to **1650763,73** wavelengths in vacuum of the radiation corresponding to the transition between the energy levels $2p_{10}$ and 5ds of the **krypton-86**.

- **Mass:-** kilogram (kg) A certain **platinum-iridium cylinder**, shall hence forth be considered to be the unit of mass, (diameter equal to its height).

- **Time:-** second (s) The duration of **9.192.631.770** periods of the radiation corresponding to transition between the two hyper fine levels of the ground state of the **cesium – 133 atom**

The Second Lecture

Dimensions and Dimensional equations

DIMENSIONS AND DIMENSIONAL EQUATIONS

The dimensions of physical quantity are the powers to which the fundamental units of length, mass, and time are raised to the unit of the given quantity. **The equation**:-

$$[Q] = L^{\pm a} M^{\pm b} T^{\pm c}$$

Which states the relation between the unit of a given quantity and the basic units is called a **dimensional equation**.

The dimensional equations can be derived from the equatic representing the between physical quantity, **for examples:-**

1) Area of rectangle = length × breadth

$$[\text{Area}] = [L]\ [L]\ [M^0]\ [T^0] = L^2$$

2) Volume of a cube = Length × Breadth × Height

$$[\text{Volume}] = [L] \times [L] \times [L] = L^3$$

3) Velocity or speed $= \dfrac{[distance]}{[time]} = \dfrac{L}{T} = L\,T^{-1}$

4) Acceleration $= \dfrac{[velocity]}{[time]} = \dfrac{[L]}{[T] \times [T]} = L\,T^{-2}$

5) Force = Mass × Acceleration

$$[F] = [M]\ [LT^{-2}] = L\ M\ T^{-2}$$

6) Work = Force × Distance

$[W] = [L\ M\ T^{-2}]\ [L] = L^2\ M\ T^{-2}$

7) Kinetic Energy $= \frac{1}{2}\ mass \times (velocity)^2$

$[K.E] = [Mass][velocity]^2 = [M]\ [L\ T^{-1}]^2 = M\ L^2\ T^{-2}$

8) Power = Work / Time

$[P] = [W] / [T] = \frac{[L^2 M\ T^{-2}]}{[T]} = L^2\ M\ T^{-3}$

9) Pressure = Force / Area

$[P] = [F] / [A] = \frac{[L\ M\ T^{-2}]}{[L^2]} = L^{-1} M\ T^{-2}$

10) [Stress] = [Force / Area]

$[S] = L^{-1}\ M\ T^{-2}$

Uses of Dimensional Equations

The following examples are the main uses of dimensional equations:-

1) Physical equation must be dimensionally homogeneous; **for example:-**

$$V^2 = V_o^2 + 2\,a\,x$$

The dimension of each term, we have:-

$$[V^2] = L^2\,T^{-2}$$

And ; $[V_o^2 + 2\,a\,x] = L^2\,T^{-2} + L\,T^{-2} \times L = 2\,L^2\,T^{-2} = L^2\,T^{-2}$

Thus, we find that all the terms have identical dimensional formula

2) To check the accuracies of physical equation; **for example:-**

$$t = 2\pi\sqrt{\frac{K^2 + L^2}{L\,g}}$$

The relation can be rewritten in the form

$$t^2 = 4\pi^2\left(\frac{K^2}{L\,g} + \frac{L^2}{L\,g}\right)$$

Now $$[t^2] = T^2$$

And $$\left[\frac{K^2}{L\,g}\right] = \frac{L^2}{L \times L\,T^{-2}} = \frac{L^2}{L^2\,T^{-2}} = T^2$$

And

$$\left[\frac{L}{g}\right] = \frac{L}{L\,T^{-2}} = T^2$$

Hence, the equation is **correct.**

To change from one system of unit to another; it has been shown that:-

$$n_1[u_1] = n_2[u_2]$$

Where:-

(n_1) and **(n_2)** are the measure number of given physical quantities in terms of the absolute units (**u_1**) and **(u_2)** respectively.

If (L^a M^b T^c) is the dimensional formula of the quantity, (L_1 M_1 T_1) and (L_2 M_2 T_2), basic units of a two system.

Then; $[u_1] = (L_1^a\ M_1^b\ T_1^c)$ and $[u_2] = (L_2^a\ M_2^b\ T_2^c)$

Which gives:-

$$n_2 = n_1 \left(\frac{L_1}{L_2}\right)^a \left(\frac{M_1}{M_2}\right)^b \left(\frac{T_1}{T_2}\right)^c$$

Where **(n_2)** can be found out, if **(n_1)** is given, thus the knowledge of dimensions of physical quantity enables us to convert the measure number from one system of units to that in another.

Example:-

If a given force be 1 pound. Calculate its measure in dynes.

$[F] = L\ M\ T^{-2}$

a = 1 ; b = 1 ; and c = - 2

$$n_2 = n_1 \left(\frac{L_1}{L_2}\right)^a \left(\frac{M_1}{M_2}\right)^b \left(\frac{T_1}{T_2}\right)^c$$

$$n_2 = 1 \left(\frac{ft}{cm}\right)^1 \left(\frac{Ib}{gm}\right)^1 \left(\frac{s}{s}\right)^{-2}$$

$$n_2 = 1\ (30.43)^1 (453.6)^1 \left(\frac{1}{1}\right)^{-2} = 13803.048\ (dynes)$$

Example:- The value of acceleration due to gravity is (32 ft/ sec^2). What is its value if the unit of the length is the mile and that of time is minute.

Solution:- We know that [acceleration] = $LM^{o}\ T^{-2} = L\ T^{-2}$

a = 1 ; b = 0 ; and c = - 2

$$n_2 = n_1 \left(\frac{L_1}{L_2}\right)^a \left(\frac{M_1}{M_2}\right)^b \left(\frac{T_1}{T_2}\right)^c$$

$$n_2 = 32 \left(\frac{foot}{mile}\right)^1 \left(\frac{Ib}{gm}\right)^0 \left(\frac{s}{min}\right)^{-2}$$

mile = 5280 foot ; 1 mint = 60 sec

$$n_2 = 32 \left(\frac{1}{5280}\right)^1 \left(\frac{1}{60}\right)^{-2} = \frac{32 \times 60 \times 60}{5280}$$

$$n_2 = 21.82\ (mile.mint^{-2})$$

Example:- Find the number of watts in one horse power.

(given that 1 Ib =453.6 gm, and 1 ft=30.48 cm, and g=32 ft/sec^2).

Solution:- $[P] = \frac{W}{t} = \frac{F.x}{t} = \frac{(m.g).x}{t} = \frac{\left(m.\frac{x}{t^2}\right).x}{t} = \frac{m.x^2}{t^3} = m.x^2.t^{-3}$

[power]= $L^2 M T^{-3}$ $\Longrightarrow$ a = 2 ; b = 1 ; and c = - 3

and n_1=1 (horse power)

n_1=550 × 32 (foot pounds)

$$n_2 = 550 \times 32 \left[\frac{ft}{cm}\right]^2 \left[\frac{Ib}{gm}\right]^1 \left[\frac{s}{s}\right]^{-3} \left(\frac{ergs}{\sec}\right)$$

$$n_2 = 550 \times 32\,(30.48)^2(453.6)$$

$$n_2 = 746.4 \times 10^7 \left(\frac{ergs}{sec}\right) = 746.4\,(Watts)$$

Example:-

The velocity with which a transverse wave travels along a stretched string may depend upon; (i) stretching force; (ii) the mass of the string (M); (iii) its length (L).

Solution:-

Let be write:- $\nu \propto F^{a}\, M^{b}\, L^{c}$

Or $\nu = k\, F^{a}\, M^{b}\, L^{c}$ (k; constant)

$\nu = \frac{x}{t}$; $[\nu] = L^{1}\, M^{0}\, T^{-1}$

From dimensional equation, we get:-

$$L^{1}\, M^{0}\, T^{-1} = (L\, M\, T^{-2})^{a}\, M^{b}\, L^{c}$$

Or $$L^{1}\, M^{0}\, T^{-1} = L^{a+c}\, M^{a+b}\, T^{-2a}$$

$$L^1\, M^0\, T^{-1} = L^{a+c}\, M^{a+b}\, T^{-2a}$$

Equating the powers *L, M, and T* on two sides, we get:-

$$a + c = 1 \quad ; \quad a + b = 0 \quad ; and \quad -2a = -1$$

$$a = \frac{1}{2} \quad ; \quad c = \frac{1}{2} \quad ; \quad b = -\frac{1}{2}$$

Hence, $\nu = k\, F^{\frac{1}{2}}\, M^{-\frac{1}{2}}\, L^{\frac{1}{2}}$

Or $\nu = k\sqrt{\frac{F\,L}{M}}$

Home work (H.W)

Q1/ In engineering work found that the volume (V) of water which point of a canal during (t) second is connected with the cross-section (A) of the canal and velocity (v) of the water by the relation: ($V = k\ v\ A\ t$). Test by the method of dimensions if the relation is correct or not.

Q2/ Calculate:- (i) number of dynes in a newton?

(ii) number of ergs in a Joule?

(iii) number of watts in a horse power?

(Ans.:- 10^5 , 10^7 , 746)

Q3/ Convert 4.2 Joules into foot-poundals.

(Ans.:- 1 (J) = 0.737 foot-pond)

Q4/ Prove dimensionally that the velocity acquired by a body after a free fall through a vertical height (h) is given by,

$$V^2 = k\,g\,h$$

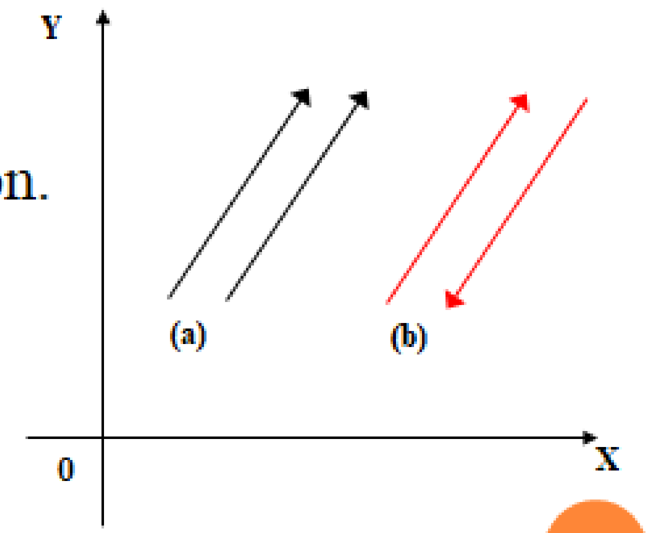
The Third Lecture

Vectors

Vectors

Concept of direction

When we are given a straight line, we can move a long it in two opposite senses; these are distinguished by assigning to each a sign, plus or minus, we say that the line is oriented and call it an axis. The coordinate axis X and Y and oriented lines in which the positive senses are as indicated in figure



An oriented line or axis defines a direction.

a) Parallel direction.

b) Antiparallel direction.

c) Oriented coordinate axis

Vector and Scalar Quantity

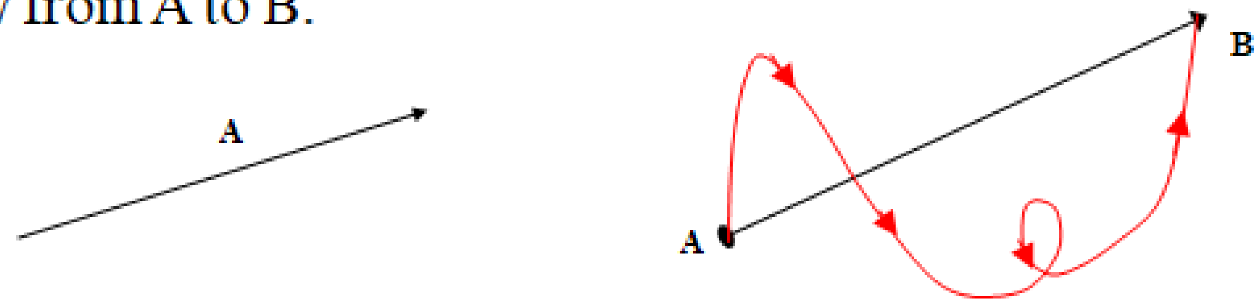
In our study of physics, we often need to work with physical quantities that have both numerical and directional properties.

A scalar quantity is completely specified by a single value with an appropriate unit and has no direction, for examples (temperature, volume, mass, speed, work, energy, and time intervals).

A vector quantity is completely specified by a number and appropriate units plus direction, for example (velocity, acceleration, force, and torque).

Representative of Vectors

If a particle moves from point A to point B a long straight path as shown in Figure. We represent this displacement by drawing an arrow from A to B.

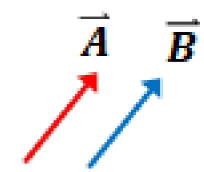


The direction of the arrowhead represents the direction of the displacement, and the length of the arrow represents the magnitude of the displacement or displacement depends only on the initial and final positions, so the displacement vector is independent of the path taken by the particle between these two points. Also, we can use a boldface letter with an arrow over the letter. Such as, $(\vec{A})$ to represent vector (A).

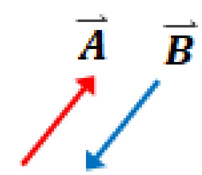
The magnitude of the vector $(\vec{A})$ is written either *(A)* or $|\vec{A}|$; The magnitude of a vector has physical units, such as meters for displacement. The magnitude of a vector is always a positive number.

Some properties of vector:-

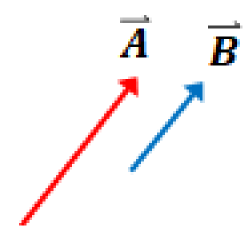
1) If two vectors $(\vec{A}\ and\ \vec{B})$ are equal in magnitudes and in same direction and parallel, we say that:- $\vec{A} = \vec{B}$



2) If two vectors are equal in magnitude and in opposite direction with each to other :- $\vec{A} = -\vec{B} \quad \Longrightarrow or \quad \vec{B} = -\vec{A}$



3) If two vectors in parallel with each to other, and not equals in magnitude then :- $\vec{A} = \lambda\,\vec{B}$; (λ: ratio between $\vec{A}$ & $\vec{B}$)



Addition of Vectors

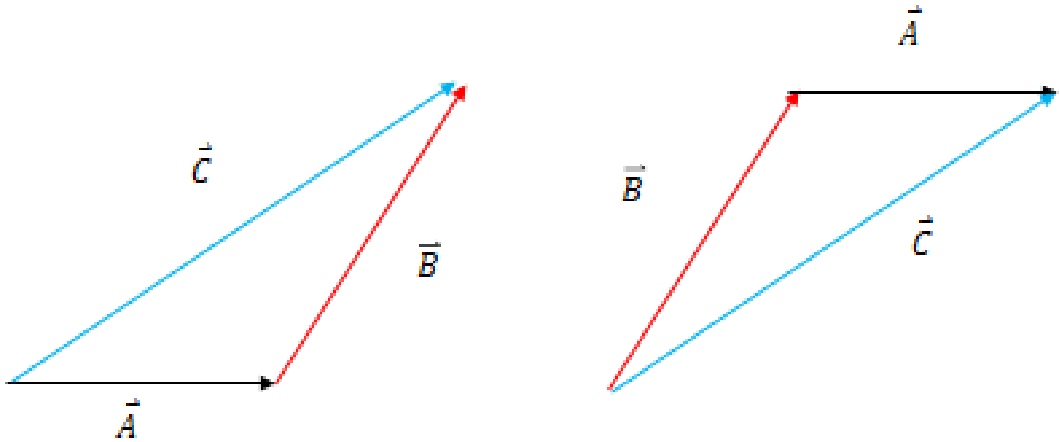
To find sun of two vectors ($\vec{A}$ & $\vec{B}$) added; the graphical method used. Therefore:-

1) Draw an arrow to represent vector ($\vec{A}$). The arrow points in the direction of the vector ($\vec{A}$). The value of a vector is not changed by moving it; as long as its direction and magnitude is not changed.

2) Draw the second vector arrow starting where the first ends; or (tail of the second arrow at the tip of the first).

3) Draw an arrow starting from the tail of the first and ending at the tip of the second. This arrow represents the sum of the two vectors as in figures:-

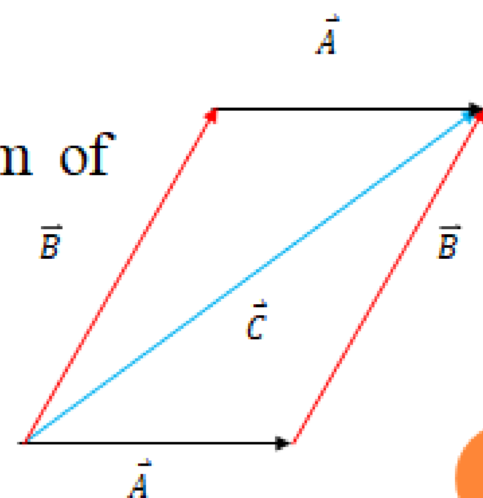
$\vec{C} = \vec{A} + \vec{B}$ ($\vec{C}$: sum of two vectors $\vec{A}$ *and* $\vec{B}$)



The result being the same if the order in which the vectors ($\vec{B}$ & $\vec{A}$) are added is reversed, or

$$\vec{C} = \vec{B} + \vec{A}$$

Figure shows that the vector ($\vec{C}$) is the sum of vectors $\vec{A}$ and $\vec{B}$, then:-



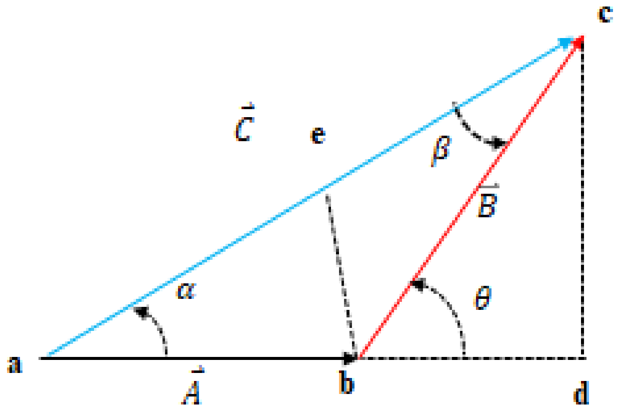
$$\vec{C} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

Thus, vector addition is commutative.

Example:-

A man walks (320 m) due east. He then continues walking along a straight line but in a different direction, and stop. Exactly (200 m) north east of his starting point. How far did he walk during the second position of the trip and in what direction?

To compute the magnitude of $\vec{C}$



From the triangle (a d c)

$$(a\,c)^2 = (a\,d)^2 + (d\,c)^2$$

But;

$$a\,d = a\,b + b\,d$$

$$a\,d = \vec{A} + \vec{B}\cos\theta$$

And

$$d\,c = \vec{B}\sin\theta$$

Therefore:- $\vec{C}^2 = (\vec{A} + B\ cos\theta)^2 + (\vec{B}\ sin\theta)^2$

$$\vec{C}^2 = \vec{A}^2 + 2\ A\ B\ cos\theta + \ B^2 cos^2\theta + \vec{B}^2\ sin^2\theta$$

$$\vec{C}^2 = \vec{A}^2 + 2\ A\ B\ cos\theta + \ B^2\ (cos^2\theta + sin^2\theta)$$

$$\vec{C}^2 = \vec{A}^2 + 2\ A\ B\ cos\theta + \ B^2$$

$$\vec{C} = \sqrt{\vec{A}^2 + \ B^2 + 2\ A\ B\ cos\theta}$$

To determine the direction of $(\vec{C})$, we need only find the angle (α).

From the figure:- the triangle (a b c)

$sin\alpha = \frac{c\ d}{a\ c} = \frac{c\ d}{\vec{C}} \quad \Rightarrow \quad cd = \vec{C}\ sin\alpha$ .... (1)

And in triangle (b d c)

$sin\theta = \frac{c\ d}{b\ c} = \frac{c\ d}{\vec{B}} \quad \Rightarrow \quad cd = \vec{B}\ sin\theta$ ....(2)

Therefore:- (from eq. (1) and eq. (2))

$$\vec{C}\ sin\alpha = \ \vec{B}\ sin\theta$$

Or; $= \frac{\vec{C}}{sin\theta} = \frac{\vec{B}}{sin\alpha}$ ...... **(3)**

Similarly:- in triangle (a b c)

$$sin\alpha = \frac{b\ e}{a\ b} = \frac{b\ e}{\vec{A}}$$

$b\ e = \overrightarrow{A}\, sin\alpha$ ...... **(4)**

And in triangle (b e c)

$$sin\beta = \frac{b\ e}{b\ c} = \frac{b\ e}{\vec{B}}$$

Or, $b\ e = \vec{B}\ sin\beta$ ......**(5)**

From eq. (4) and eq. (5), find:-

$$\vec{A}\ sin\alpha = \vec{B}\ sin\beta$$

Or; $\frac{\vec{B}}{sin\alpha} = \frac{\vec{A}}{sin\beta}$ ……(6)

Combining both equations (3 and 6), we get:-

$\frac{\vec{C}}{sin\theta} = \frac{\vec{A}}{sin\beta} = \frac{\vec{B}}{sin\alpha}$ Law of sine

In special case when ($\vec{A}$) and ($\vec{B}$) are perpendiculars ($\theta = 90^o$) the following relations hold:-

$$\vec{C} = \sqrt{\vec{A}^2 + \vec{B}^2}$$

And, $tan\alpha = \frac{\vec{B}}{\vec{A}}$

$$\alpha = tan^{-1}\frac{\vec{B}}{\vec{A}}$$

Vector Subtracted (D)

The difference between two vectors ($\vec{A}$ *and* $\vec{B}$) is obtained by adding to the first the negative (or opposite) of the second:- or

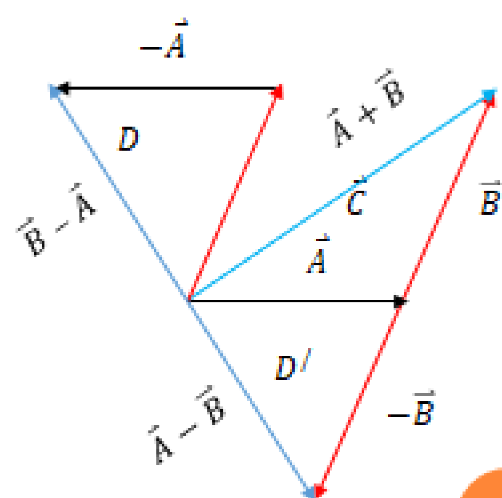
$$D = \vec{A} - \vec{B}$$

And

$$\vec{A} - \vec{B} \neq \vec{B} - \vec{A}$$

$$D \neq D^{/}$$

As shown in Figure;



Therefore:-

Vector difference is anti commutative.

The magnitude of the difference (D) is:-

$$D = \sqrt{\vec{A}^2 + \vec{B}^2 - 2\,\vec{A}\,\vec{B}\,cos\theta}$$

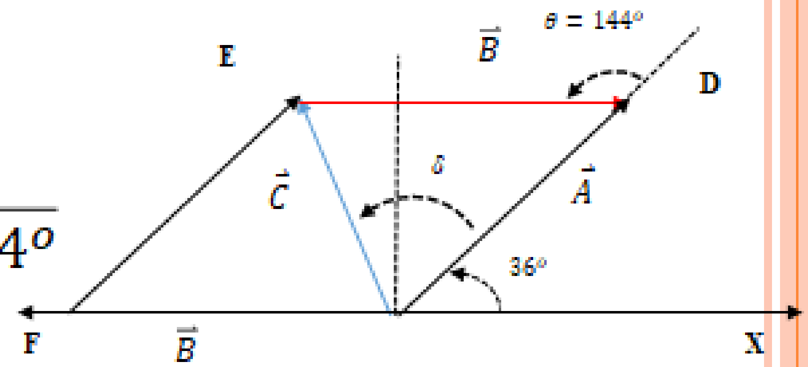
Example:-

Given two vectors:- is 6 units long and makes an angle of with the positive (x-axis); is 7 units long and is in the direction of the negative (x-axis); find:- (a) the sum of the two vectors, (b) the difference between the two vectors.

a) In triangle ( )

$$\theta = 180^o - 36^o = 144^o$$

$$\vec{C} = \sqrt{36 + 49 + 2\,(6)(7)cos144^o}$$
$$= 4.128$$



To find the angle between $\vec{C}$ and $\vec{A}$, then:-

$$\frac{\vec{C}}{sin\theta} = \frac{\vec{B}}{sin\delta} \quad \Longrightarrow \quad \frac{4.128}{sin144} = \frac{7}{sin\delta}$$

$$sin\delta = 0.996 \quad \Longrightarrow \quad \delta = 85^o$$

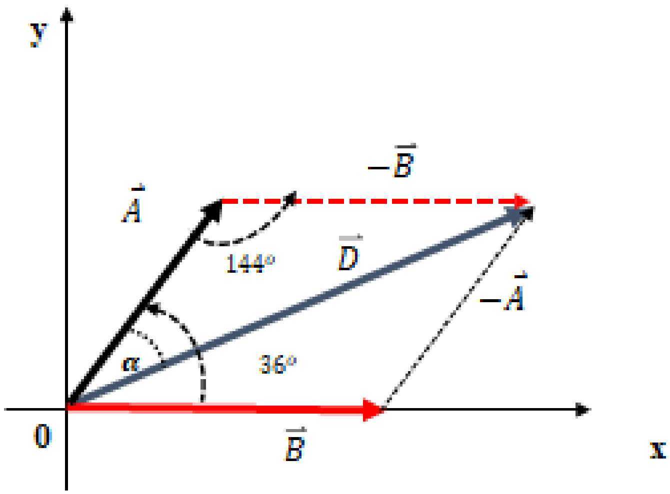
A direction $(36^o + 85 = 121^o)$ with + x,

a) To find the difference

$$D = \vec{A} - \vec{B}$$

$$\because D = \sqrt{36 + 49 - 2\,(6)(7)cos144^o} = 12.31\ (units)$$

To find the direction of (D).



$$\frac{D}{sin144^o} = \frac{\vec{B}}{sin\alpha}$$

$$\frac{12.31}{sin144^o} = \frac{7}{sin\alpha}$$

$$sin\alpha = 0.334 \quad \Longrightarrow \quad \alpha = 19.5^o$$

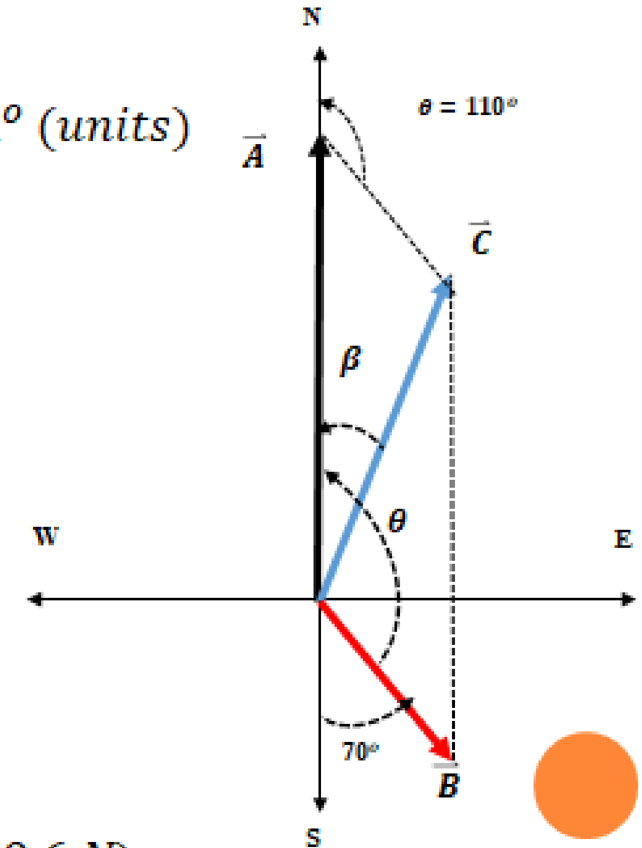
Direction of is $\vec{D}$ is $(36^o - 19.5^o) = 16.5^o$ in + X

Example:-

Vector ($\vec{A} = 15\ (units)$) long in direction of North, vector ($\vec{B} = 5\ (units)$) in direction of ($S\ 70^o\ E$). Find sum of ($\vec{C} = \vec{A} + \vec{B}$)

$$\theta = 180^o - 70^o = 110^o$$

$$\vec{C} = \sqrt{(15)^2 + (5)^2 + 2\,(15)(5)\cos 110^o} = 14.1^o\ (units)$$



To obtain direction of $\vec{C}$.

$$\frac{\vec{C}}{\sin\theta} = \frac{\vec{B}}{\sin\beta} \quad \Rightarrow \quad \frac{14.1}{\sin 110} = \frac{5}{\sin\beta}$$

$$\sin\beta = \frac{5 \times 0.93969}{14.1} = 0.333$$

$$\beta = \sin^{-1} 0.333 = 19.4^o$$

Thus, the resultant motion is in the direction

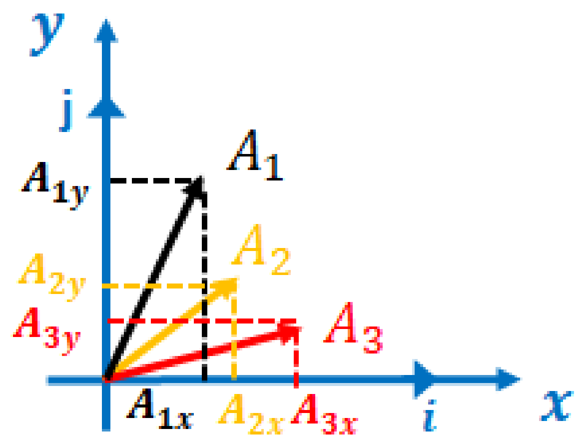
$(N\ 19.4^o\ E\ \ ;or\ \ 90 - 19.4 = 70.6 \ \Rightarrow\ E\ 70.6\ N)$

The Fourth Lecture

Addition of Several Vectors

Addition of Several Vectors

To add several vectors (A_1, A_2, A_3, ….) by using the method of components. For simplicity let us consider the case where all vectors are in one plane so that we need to use only two components. Then



$$\vec{A} = (A_{1x}\hat{\imath} + A_{1y}\hat{\jmath}) + (A_{2x}\hat{\imath} + A_{2y}\hat{\jmath}) + (A_{3x}\hat{\imath} + A_{3y}\hat{\jmath}) + \cdots + (A_{nx}\hat{\imath} + A_{ny}\hat{\jmath})$$

$$\vec{A} = (A_{1x} + A_{2x} + A_{3x} + \cdots + A_{nx})\hat{\imath} + (A_{1y} + A_{2y} + A_{3y} + \cdots + A_{ny})\hat{\jmath}$$

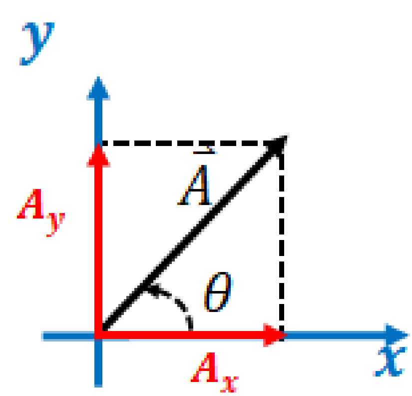
Addition of Several Vectors

Where:-

$$A_{1x} + A_{2x} + A_{3x} + \cdots + A_{nx} = \sum_{i=1}^{n} A_{ix}$$

$$A_{1y} + A_{2y} + A_{3y} + \cdots + A_{ny} = \sum_{i=1}^{n} A_{iy}$$

$$\vec{A} = \left(\sum_{i=1}^{n} A_{ix}\right)\hat{\imath} + \left(\sum_{i=1}^{n} A_{iy}\right)\hat{\jmath}$$



$$\cos\theta = \frac{A_x}{A} \Rightarrow A_x = A\cos\theta \quad ; \sin\theta = \frac{A_y}{A} \Rightarrow A_y = A\sin\theta$$

$$\vec{A} = \left(\sum_{i=1}^{n} A\cos\theta_n\right)\hat{\imath} + \left(\sum_{i=1}^{n} A\sin\theta_n\right)\hat{\jmath}$$

Addition of Several Vectors

The magnitude of $(\vec{A})$ is:-

$$\vec{A} = \sqrt{(A_{ix})^2 + (A_{iy})^2}$$

And direction of $(\vec{A})$ is:-

$$tan\theta = \frac{A_{iy}}{A_{ix}} \quad \Longrightarrow \quad \theta = tan^{-1}\frac{A_{iy}}{A_{ix}}$$

Examples

1) If the vector $(\vec{A} = 2\hat{\imath} + 3\hat{\jmath} - 4\hat{k})$

Find $\hat{u}(A); |\hat{u}(A)|$

$$\hat{u}(A) = \frac{\vec{A}}{|A|}$$

$$|A| = \sqrt{(2)^2+(3)^2+(4)^2} = \sqrt{29}$$

$$\hat{u}(A) = \frac{\vec{A}}{|A|} = \frac{2i + 3j - 4k}{\sqrt{29}} = \frac{2}{\sqrt{29}}i + \frac{3}{\sqrt{29}}j - \frac{4}{\sqrt{29}}k$$

$$|\hat{u}(A)| = \left[\left(\frac{2}{\sqrt{29}}\right)^2 + \left(\frac{3}{\sqrt{29}}\right)^2 + \left(\frac{-4}{\sqrt{29}}\right)^2\right]^{\frac{1}{2}}$$

$$|\hat{u}(A)| = \left[\frac{4}{29} + \frac{9}{29} + \frac{16}{29}\right]^{\frac{1}{2}} = 1$$

The Fifth Lecture

Vector Multiplication

Vector Multiplication

Two kinds of product:-

1. Scalar product (or dot product)
2. Vector product (or cross product)

Scalar product:- the scalar product of two vectors $(\vec{A})$ and $(\vec{B})$, denoted by $\vec{A}.\vec{B}$ (read $\vec{A}$ dot $\vec{B}$) is defined as the scalar quantity obtained or:-

$$\vec{A}.\vec{B} = A\ B\ cos\theta$$

$$\vec{A}.\vec{B} = |A|\ |B|\ cos\theta$$

$$cos\theta = \frac{\vec{A}.\vec{B}}{|A||B|}$$

Vector Multiplication

- If $\vec{A} \,//\, \vec{B}$ or $\theta = 0$; $cos0 = 1$

$$\therefore\ \vec{A}\,.\vec{B} = A\,B$$

- If $\vec{A} = \vec{B}$ then

$$\vec{A}\,.\vec{B} = A^2 = B^2$$

- If $\vec{A} \perp \vec{B}$ or $\theta = 90^o$ $\Longrightarrow$ $cos90 = 0$

$$\therefore\ \vec{A}\,.\vec{B} = 0$$

- $\vec{A}\,.\vec{B} = \vec{B}\,.\,\vec{A}$ **is commutative**.

Example

If a vector $\vec{A} = A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}$ and vector $\vec{B} = B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k}$; find the dot product of $(\vec{A}\,.\,\vec{B}\,)$.

$$\vec{A}\,.\,\vec{B} = \left(A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}\right).\,(B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k})$$

$$\vec{A}\,.\,\vec{B} = A_xB_x(\hat{\imath}.\hat{\imath}) + A_xB_y(\hat{\imath}.\hat{\jmath}) + A_xB_z\left(\hat{\imath}.\hat{k}\right) + A_yB_x(\hat{\jmath}.\hat{\imath}) + A_yB_y(\hat{\jmath}.\hat{\jmath}) + A_yB_z\left(\hat{\jmath}.\hat{k}\right) + A_zB_x\left(\hat{k}.\hat{\imath}\right) + A_zB_y\left(\hat{k}.\hat{\jmath}\right) + A_zB_z\left(\hat{k}.\hat{k}\right)$$

$\because\ \hat{\imath}.\hat{\imath} = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k} = 1\,;\quad \theta = 0 \quad \Longrightarrow \quad cos0 = 1$

And $\hat{\imath}.\hat{\jmath} = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath} = 0\ \ ;\theta = 90^o \quad \Longrightarrow \quad cos90 = 0$

$$\vec{A}\,.\,\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

To find the angle between vector $\left(\vec{A}\ and\ \vec{B}\right)$.

$$cos\theta = \frac{A_xB_x + A_yB_y + A_zB_z}{|\vec{A}||\vec{B}|}$$

Example

Find the angle between the vector ($\vec{A}$) and vector ($\vec{B}$). Where;

$$\vec{A} = 2\,\hat{\imath} + 3\,\hat{\jmath}$$

$$\because\ cos\theta = \frac{\vec{A}\,.\,\vec{B}}{|\vec{A}||\vec{B}|}$$

$$\vec{A}\,.\,\vec{B} = A_xB_x + A_yB_y + A_zB_z$$

$$\vec{A}\,.\,\vec{B} = (2)(-1) + (3)(1) + (-1)(2) = -1$$

$$|\vec{A}| = \sqrt{4+9+1} = 3.74$$

$$|\vec{B}| = \sqrt{1+1+4} = 2.45$$

$$cos\theta = \frac{-1}{(3.74).\,(2.45)}$$

$$\theta = cos^{-1}(-0.109) = 96.3^o$$

Vector Product

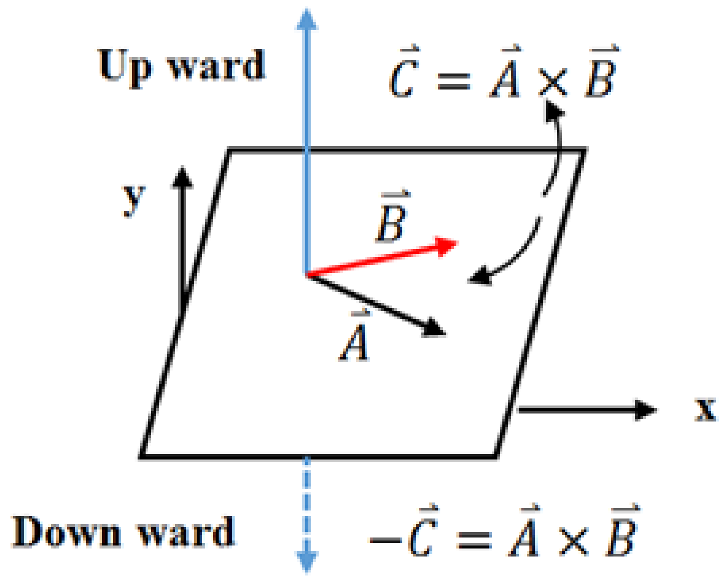
i. The vector product of two vectors ($\vec{A}$ *and* $\vec{B}$) is denoted by ($\vec{A} \times \vec{B}$) (also called the cross product); the two vectors then lie in a plane.

ii. The vector products is defined as a vector quantity with a direction perpendicular to this plane (to both $\vec{A}$ *and* $\vec{B}$) and a magnitude given by ($A\,B\,sin\theta$) or :-

If $\vec{C} = \vec{A} \times \vec{B}$ then $\vec{C} = A\,B\,sin\theta \quad \hat{u}$

Iii. The product of two vectors is a vector the direction determined by the "right hand rule". The curl fingers of the right hand a round this perpendicular line so that the tum b then gives the direction of the vector product, as in Fig.:-

Vector Product



The right hand product is $\vec{A} \times \vec{B} = A\,B\,sin\theta\ \hat{u}$

Where $(\hat{u})$ is unit vector , which is indicate to direction.

From the Fig. :- that

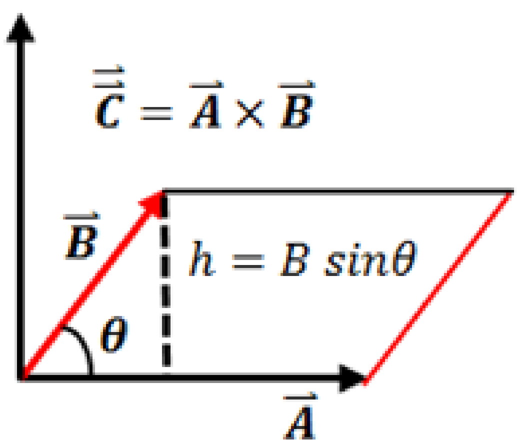
$\vec{A} \times \vec{B} \neq -\vec{B} \times \vec{A}$ (vector product is no commutative)

Vector Product

iv. If $\vec{A} = \vec{B}$,

Or if ($\vec{A}$) parallel to ($\vec{B}$) then ($sin\theta = 0$) ; and ($\vec{A} \times \vec{B} = 0$)

v. Vector product also uses to find the area of a parallelogram with sides ($\vec{A}$) and ($\vec{B}$) or $|\vec{A} \times \vec{B}|$ = the area of a parallelogram. From the Fig.:



Vector Product

$$\vec{A} \times \vec{B} = A\,B\,sin\theta \;\; \hat{u}$$

The magnitude is $|\vec{A} \times \vec{B}| = A\,B\,sin\theta$

$$|\vec{A} \times \vec{B}| = A\,h = Area$$

$$h = B\,sin\theta$$

The component vector product of $(\vec{A})$ and $(\vec{B})$ are known:-

$$\vec{A} \times \vec{B} = (A_x\,\hat{\imath} + A_y\,\hat{\jmath} + A_z\,\hat{k}) \times (B_x\,\hat{\imath} + B_y\,\hat{\jmath} + B_z\,\hat{k})$$

$$\vec{A} \times \vec{B} = A_x\,\hat{\imath} \times B_x\,\hat{\imath} + A_x\hat{\imath} \times B_y\,\hat{\jmath} + A_x\hat{\imath} \times B_z\,\hat{k} + A_y\hat{\jmath} \times B_x\,\hat{\imath}$$
$$+ A_y\,\hat{\jmath} \times B_y\,\hat{\jmath} + A_y\hat{\jmath} \times B_z\hat{k} + A_z\hat{k} \times B_x\hat{\imath} + A_z\hat{k} \times B_y\hat{\jmath}$$
$$+ A_z\hat{k} \times B_z\hat{k}$$

$$\vec{A} \times \vec{B} = (A_yB_z - A_z\,B_y)\,\hat{\imath} + (A_zB_x - A_xB_z)\,\hat{\jmath} + (A_xB_y - A_yB_x)\hat{k}$$

Vector Product

Where:-

$\hat{\imath} \times j = k \quad \Longrightarrow \quad j \times \hat{\imath} = -k$

$$\hat{\jmath} \times k = i \quad \Longrightarrow \quad k \times \hat{\jmath} = -i$$

$$k \times i = j \quad \Longrightarrow \quad i \times k = -j$$

And

$$\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = 0$$

If $(\vec{C} = \vec{A} \times \vec{B})$; the component of $(\vec{C})$ is:-

$$C_x = A_y B_z - A_z B_y \quad ; \quad C_y = A_z B_x - A_x B_z \; ;$$

$C_z = A_x B_y - A_y B_x$

Vector Product

The same result obtained by using determinant from:-

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= +\hat{\imath}\left(A_y B_z - A_z B_y\right) - j\left(A_x B_z - A_z B_x\right) + k\left(A_x B_y - A_y B_x\right)$$

Example:- If $(\vec{A} = 3\,\hat{\imath} - \hat{\jmath} + 2\,\hat{k})$ **and** $\vec{B} = 2\,\hat{\imath} + 3\,\hat{\jmath} - \hat{k}$ **; Find:-**

1. $\vec{A} \times \vec{B}$

$$|\vec{A} \times \vec{B}| = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix} = i\begin{vmatrix} -1 & 2 \\ 3 & -1 \end{vmatrix} - j\begin{vmatrix} 3 & 2 \\ 2 & -1 \end{vmatrix} + k\begin{vmatrix} 3 & -1 \\ 2 & 3 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = -5\,\hat{\imath} + 7\,\hat{\jmath} + 11\,\hat{k}$$

Vector Product

2. The Angle between $(\vec{A})$ and $(\vec{B})$

$$\boldsymbol{cos\theta = \frac{A_x B_x + A_y B_y + A_z B_z}{|\vec{A}||\vec{B}|}}$$

$$|\vec{\boldsymbol{A}}| = \sqrt{(3)^2 + (-1)^2 + (2)^2} = \sqrt{14}$$

$$|\vec{\boldsymbol{B}}| = \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$$

$$cos\theta = \frac{6 - 3 - 2}{(\sqrt{14})(\sqrt{14})} = \frac{1}{14} = 0.072$$

$$\theta = cos^{-1}(0.072) = 85.87^o$$