Moving Reference coordinate systems
حركة المحاور المرجعية

جامعة الموصل
كلية التربية للعلوم الصرفة
قسم الفيزياء

المادة الميكانيك التحليلي
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Translation of coordinate systems

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Accelerated Coordinate Systems and Inertial Forces:

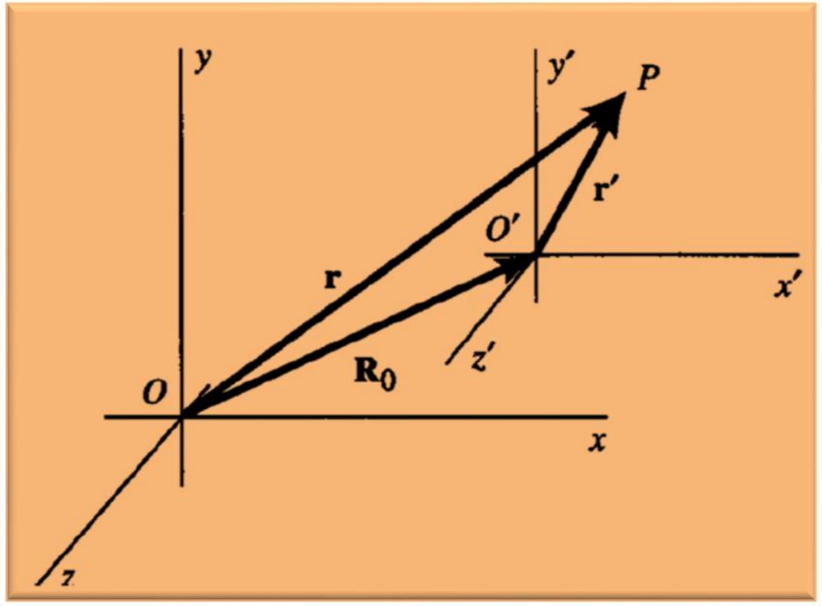
Consider the case of a coordinate system that undergoes pure translation. In Figure Oxyz are the primary coordinate axes (assumed fixed), and O'x'y'z' are the moving axes. In the case of pure translation, the respective axes Ox and O'x', and so on, remain parallel. The position vector of a particle P is denoted by **r** in the fixed system and by **r'** in the moving system. The displacement OO' of the moving origin is denoted by **Ro** Thus, from the triangle OO'P, we have:

$$\mathbf{r} = \mathbf{R}_0 + \mathbf{r}'$$

Taking the first and second time derivatives gives:

$$\mathbf{v} = \mathbf{V}_0 + \mathbf{v}'$$

$$\mathbf{a} = \mathbf{A}_0 + \mathbf{a}'$$



In which $\mathbf{V_o}$ and $\mathbf{A_o}$ are the velocity and acceleration of the moving system, and **v'** and **a'** are the velocity and acceleration of the particle in the moving system.

In particular, if the moving system is not accelerating, so that $A_o = 0$, then:

$$\mathbf{a} = \mathbf{a}'$$

So the acceleration is the same in either system. Consequently, if the primary system is inertial, Newton's second law **F** = m**a** becomes **F** = m**a'** in the moving system; that is, the moving system is also an inertial system (provided it is not rotating). Thus, as far as Newtonian mechanics is concerned, we cannot specify a unique coordinate system; if Newton's laws hold in one system, they are also valid in any other system moving with uniform velocity relative to the first

On the other hand if the moving system is accelerating, then Newton's second law becomes:

$$\mathbf{F} = m\mathbf{A}_0 + m\mathbf{a}'$$

or

$$\mathbf{F} - m\mathbf{A}_0 = m\mathbf{a}' \qquad (1)$$

For the equation of motion in the accelerating system. If we wish, we can write Equation (1b) in the form:

$$\mathbf{F}' = m\mathbf{a}'$$

In which **F'** = **F** + (-m**A**$_o$). That is, an acceleration **A**$_o$ of the reference system can be taken into account by adding an **inertial term** –m**A**$_o$ to the force F and equating the result to the product of mass and acceleration in the moving system. Inertial terms in the equations of motion are sometimes called **inertial forces**, or **fictitious forces**. Such "forces" are not due to interactions with other bodies, rather, they stem from the acceleration of the reference system.

Example:

A block of wood rests on a rough horizontal table. If the table is accelerated in a horizontal direction, under what conditions will the block slip?

Solution:

Let μ_s be the coefficient of static friction between the block and the table top. Then the force of friction **F** has a maximum value of μ_s mg where, m is the mass of the block. The condition for slipping is that the inertial force –mAo exceeds the frictional force, where A_o is the acceleration of the table. Hence, the condition for slipping is:

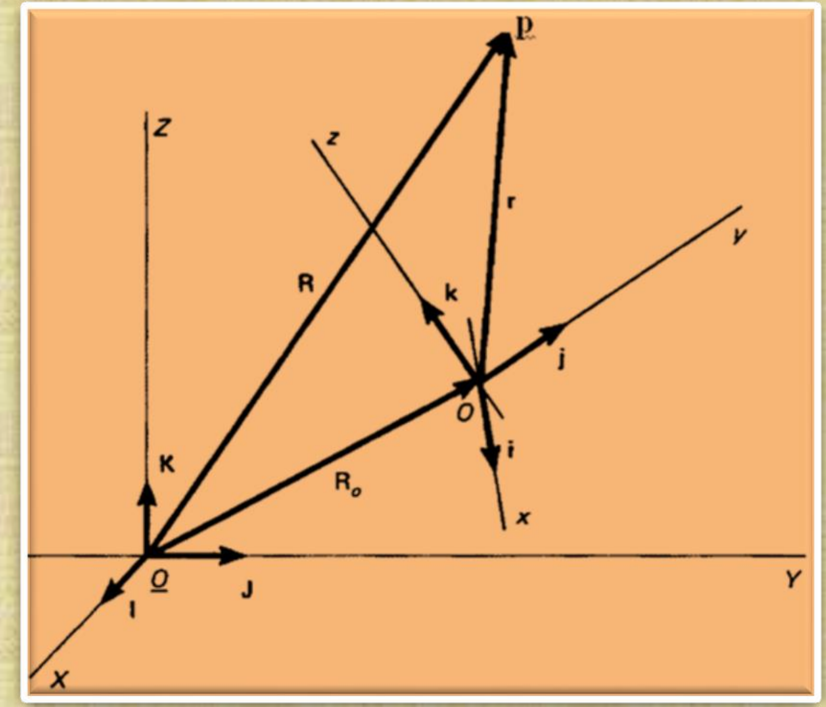
$$|-m\mathbf{A}_0| > \mu_s m\mathbf{g}$$

or

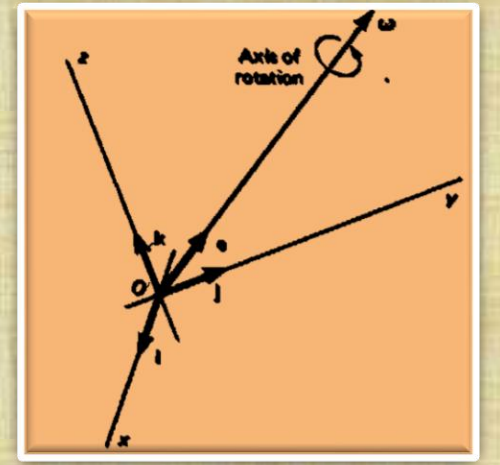
$$A_0 > \mu_s g$$

Rotating Coordinate Systems:

We consider the case in which the reference system undergoes both translation and rotation relative to the inertial system. The position vector of the particle in the inertial system is denoted by **R**, and in the moving system by **r**, as shown in figure.



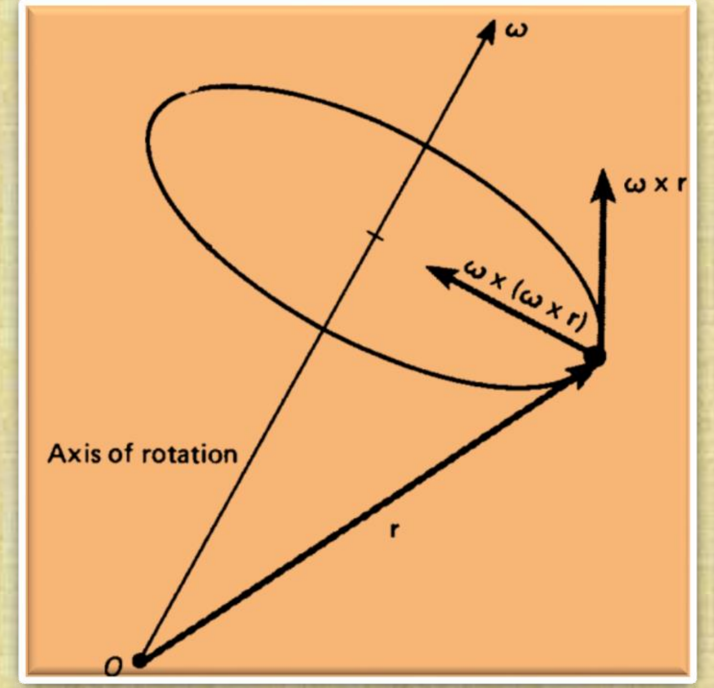
Let the direction of the axis of rotation of the moving system be specified by the unit vector **e**, as shown in figure and let ω be the angular speed about the axis:



Then the angular velocity of the moving system is:

$$\boldsymbol{\omega} = \omega \mathbf{e}$$

The velocity imparted to a particle due to rotation about an axis can be expressed by the cross product



$$\mathbf{v}_{rot} = \boldsymbol{\omega} \times \mathbf{r}$$

To find how we get $\mathbf{v}_{rot}$,we have:

$$\mathbf{R} = \mathbf{R}_0 + \mathbf{r}$$

Where $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

Then we can write

$$\mathbf{R} = \mathbf{Ro} + \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$$

Taking the first time derivative we find:

$$\frac{d\vec{R}}{dt} = \vec{V}_o + \hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}$$

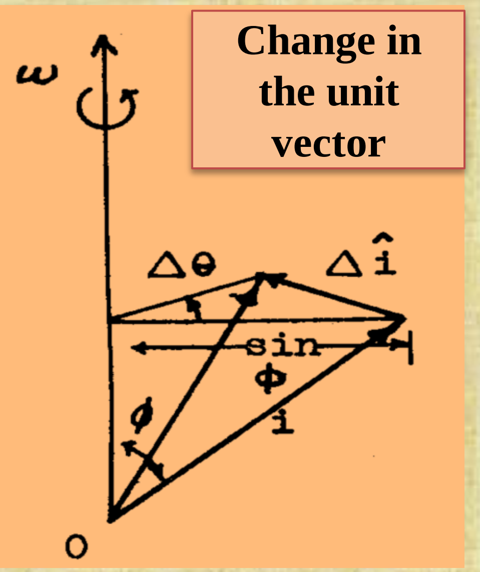
Velocity vector in the fixed system

Velocity of the particle in the moving system denoted by $\vec{r}$

Represent the velocity due to rotation of the coordinate system.

To find the time derivatives

$d\hat{k}/dt$, $d\hat{j}/dt$, $d\hat{i}/dt$ consider Figure:



From the figure we see that the magnitude

$$|\Delta\hat{i}| \simeq (\sin\phi)\ \Delta\theta$$

Where ϕ is the angle between $\hat{\imath}$ and ω. Δθ is the magnitude of rotation axes oxyz.

Let Δt be the time interval for this change. Then we can write

$$\left|\frac{d\hat{i}}{dt}\right| = \lim_{\Delta t \to 0} \left|\frac{\Delta \hat{i}}{\Delta t}\right| = (\sin \phi)\,\frac{d\theta}{dt} = \omega \sin \phi$$

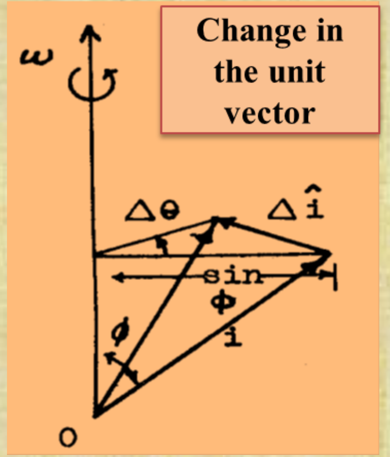
Now the direction of $\Delta\hat{\imath}$ is perpendicular to both **ω** and consequently, from the definition of the cross product, we can write:

$$\frac{d\hat{i}}{dt} = \vec{\omega} \times \hat{i}$$

Similarly, we find

$$\frac{d\hat{j}}{dt} = \vec{\omega} \times \hat{j}$$

$$\frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k}$$



Change in the unit vector

We now apply the preceding result to the last three terms in Equation as follows:

$$\frac{d\vec{R}}{dt} = \vec{V}_o + \hat{i}\dot{x} + \hat{j}\dot{y} + \hat{k}\dot{z} + x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt}$$

$$x\frac{d\hat{i}}{dt} + y\frac{d\hat{j}}{dt} + z\frac{d\hat{k}}{dt} = x(\vec{\omega} \times \hat{i}) + y(\vec{\omega} \times \hat{j}) + z(\vec{\omega} \times \hat{k})$$
$$= \vec{\omega} \times (\hat{i}x + \hat{j}y + \hat{k}z)$$
$$= \vec{\omega} \times \vec{r}$$

We can be generalized the equation $\mathbf{V} = \mathbf{v} + \mathbf{v}_{rot} + \mathbf{V}_o$ to include the rotation by writing:

$$\frac{d\mathbf{R}}{dt} = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r} + \mathbf{V}_0$$

The velocity of a moving particle measured in a primary inertial coordinate system can expressed as the sum of three vector (1) the velocity $\dot{\mathbf{r}}$ of the particle in the moving system, (2) the rotational velocity $\omega \times \mathbf{r}$ and (3) the velocity $\mathbf{V}_o$ of the origin of the moving system.

The case of any vector quantity **q**, the time derivative in the primary system (fixed) is given by adding the term $\omega \times \mathbf{q}$ to the time derivative in the rotating system, namely:

$$\left(\frac{d\mathbf{q}}{dt}\right)_{\text{fixed}} = \dot{\mathbf{q}} + \boldsymbol{\omega} \times \mathbf{q}$$

Note:

If we let q=ω, then we find $\left(\frac{d\boldsymbol{\omega}}{dt}\right)_{\text{fixed}} = \dot{\boldsymbol{\omega}}$ *since ω×ω=0*

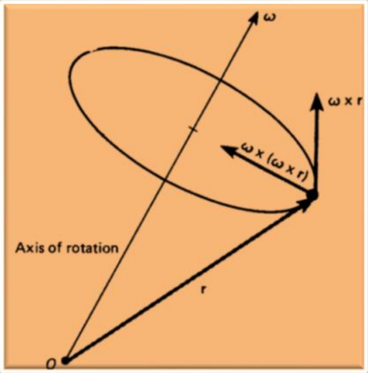
that is the angular acceleration is the same in both systems

Thus for the second derivative we find:

$$\left(\frac{d^2\mathbf{q}}{dt^2}\right)_{\text{fixed}} = \ddot{\mathbf{q}} + (\boldsymbol{\omega} \times \dot{\mathbf{q}}) + (\dot{\boldsymbol{\omega}} \times \mathbf{q}) + \boldsymbol{\omega} \times (\dot{\mathbf{q}} + \boldsymbol{\omega} \times \mathbf{q})$$

$$= \ddot{\mathbf{q}} + 2\boldsymbol{\omega} \times \dot{\mathbf{q}} + \dot{\boldsymbol{\omega}} \times \mathbf{q} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{q})$$

Use this equation to find the relationship between acceleration vectors (*hint:* **let q be equal to the quantity** $d\mathbf{R}/dt - \mathbf{V}_0 = \dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}$ **).** We then find the following result



$$\frac{d^2\mathbf{R}}{dt^2} = \ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \mathbf{A}_0$$

The first term on the right – hand side is just the acceleration of the particle in the moving system. The **next three terms** are rotational terms for the acceleration of the particle as seen in the fixed system. The **last term** is the acceleration of the moving origin.

The term $2\omega \times \dot{\mathbf{r}}$ is known as **Coriolis acceleration**. The term $\omega \times \mathbf{r}$ is the **transverse acceleration**. The last term $\omega \times (\omega \times \mathbf{r})$ is the **centripetal acceleration**. It is always directed toward the axis of rotation and is perpendicular to the axis.

General Motion of a Particle in Three Dimensions

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المحاضرة السادسة: اسئلة محلولة

Solved problems

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Problems:

1- Find the force for each of the following potential energy functions:

(a) $V = cxyz + C$

(b) $V = \alpha x^2 + \beta y^2 + \gamma z^2 + C$

(c) $V = ce^{-(\alpha x + \beta y + \gamma z)}$

(d) $V = cr^n$ in spherical coordinates

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

Solution:

(a) $\vec{F} = -\vec{\nabla} V = -\hat{i}\frac{\partial V}{\partial x} - \hat{j}\frac{\partial V}{\partial y} - \hat{k}\frac{\partial V}{\partial z}$

gradient of V

$$\vec{F} = -c\left(\hat{i}yz + \hat{j}xz + \hat{k}xy\right)$$

(b) $\vec{F} = -\vec{\nabla} V = -\hat{i}\,2\alpha x - \hat{j}\,2\beta y - \hat{k}\,2\gamma z$

(c) $\vec{F} = -\vec{\nabla} V = ce^{-(\alpha x + \beta y + \gamma z)}\left(\hat{i}\alpha + \hat{j}\beta + \hat{k}\gamma\right)$

(d) $\vec{F} = -\vec{\nabla} V = -\hat{e}_r \frac{\partial V}{\partial r} - \hat{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} - \hat{e}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$

$$\vec{F} = -\hat{e}_r cnr^{n-1}$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\sin\theta}\frac{\partial}{\partial \phi}$$

2- Which of the following forces are conservative:

(a) $\mathbf{F} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$

(b) $\mathbf{F} = \mathbf{i}y - \mathbf{j}x + \mathbf{k}z^2$

(c) $\mathbf{F} = \mathbf{i}y + \mathbf{j}x + \mathbf{k}z^3$

(d) $\mathbf{F} = -kr^{-n}\mathbf{e}_r$ in spherical coordinates

Solution: The condition for the force to be conservative can be written as:

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0$$

(a)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix} = 0$$

(b)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & z^2 \end{vmatrix} = \hat{k}(-1-1) \neq 0$$

(c)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & z^3 \end{vmatrix} = \hat{k}(1-1) = 0$$

d

$$\vec{\nabla} \times \vec{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{e}_r & \hat{e}_\theta r & \hat{e}_\phi r\sin\theta \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ -kr^{-n} & 0 & 0 \end{vmatrix} = 0$$

3- Find the value of the constant c such that each of the following forces is conservative:

(a) $\mathbf{F} = \mathbf{i}xy + \mathbf{j}cx^2 + \mathbf{k}z^3$

(b) $\mathbf{F} = \mathbf{i}(z/y) + c\mathbf{j}(xz/y^2) + \mathbf{k}(x/y)$

Solution:

$$\nabla \times \mathbf{F} = \mathbf{i}\left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right) = 0$$

(a)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & cx^2 & z^3 \end{vmatrix} = k\left(2cx - x\right)$$

$$2cx - x = 0$$

$$c = \frac{1}{2}$$

(b)

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{z}{y} & \frac{cxz}{y^2} & \frac{x}{y} \end{vmatrix}$$

$$= \hat{i}\left(-\frac{x}{y^2} - \frac{cx}{y^2}\right) + \hat{j}\left(\frac{1}{y} - \frac{1}{y}\right) + \hat{k}\left(\frac{cz}{y^2} + \frac{z}{y^2}\right)$$

$$-\frac{x}{y^2} - \frac{cx}{y^2} = 0 \qquad c = -1$$

also $\frac{cz}{y^2} + \frac{z}{y^2} = 0$ implies that $c = -1$ as it must

4- A particle of mass m moving in three dimensions under the potential energy function $V(x, y, z) = \alpha x + \beta y^2 + \gamma z^3$ has speed v_o when it passes through the origin.

(a) What will its speed be if and when it passes through the point (1,1,1)?
(b) If the point (1, 1, 1) is a turning point in the motion (v = 0), what is v_o?
(c) What are the component differential equations of motion of the particle?

Solution:

(a) $$E = \text{constant} = V(x, y, z) + \frac{1}{2}mv^2$$

at the origin $$E = 0 + \frac{1}{2}mv_o^2$$

at (1,1,1) $$E = \alpha + \beta + \gamma + \frac{1}{2}mv^2 = \frac{1}{2}mv_o^2$$

$$v^2 = v_o^2 - \frac{2}{m}(\alpha + \beta + \gamma)$$

$$v = \left[v_o^2 - \frac{2}{m}(\alpha + \beta + \gamma)\right]^{\frac{1}{2}}$$

(b) $$v_z^2 - \frac{2}{m}(\alpha + \beta + \gamma) = 0$$

$$v_z = \left[\frac{2}{m}(\alpha + \beta + \gamma)\right]^{\frac{1}{2}}$$

The potential energy function is $V(x, y, z) = \alpha x + \beta y^2 + \gamma z^3$

The component differential equations of motion of the particle

$$\mathbf{F} = \mathbf{F}_x + \mathbf{F}_y + \mathbf{F}_z$$
$$= F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$$

(c) $$m\ddot{x} = F_x = -\frac{\partial V}{\partial x}$$

$$m\ddot{x} = -\alpha$$

$$m\ddot{y} = -\frac{\partial V}{\partial y} = -2\beta y$$

$$m\ddot{z} = -\frac{\partial V}{\partial z} = -3\gamma z^2$$

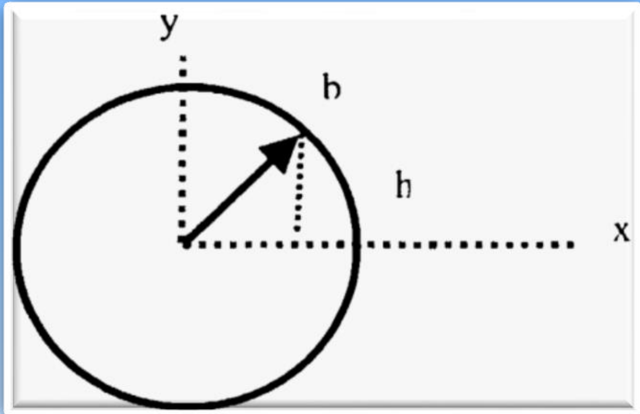
5- A particle is placed on a smooth sphere of radius b at a distance b/2 above the central plane. As the particle slides down the side of the sphere, at what point will it leave?

Solution:

$$\frac{1}{2}mv^2 + mgh = mg\frac{b}{2}$$

$$v^2 = g(b-2h)$$

$$F_r = -\frac{mv^2}{b} = -mg\cos\theta + R$$



$$\cos\theta = \frac{h}{b}$$

$$R = mg\frac{h}{b} - \frac{mv^2}{b} = \frac{mg}{b}\left[h-(b-2h)\right] = \frac{mg}{b}(3h-b)$$

the particle leaves the side of the sphere when $R = 0$

$h = \frac{b}{3}$, i.e.. $\frac{b}{3}$ above the central plane

6- A bead slides on a smooth rigid wire bent into the form of a circular loop of radius b. If the plane of the loop is vertical, and if the bead starts from rest at a point that is level with the center of the loop, find the speed of the bead at the bottom and the reaction of the wire on the bead at that point.

Solution:

$$\frac{1}{2}mv^2 + mgh = 0$$

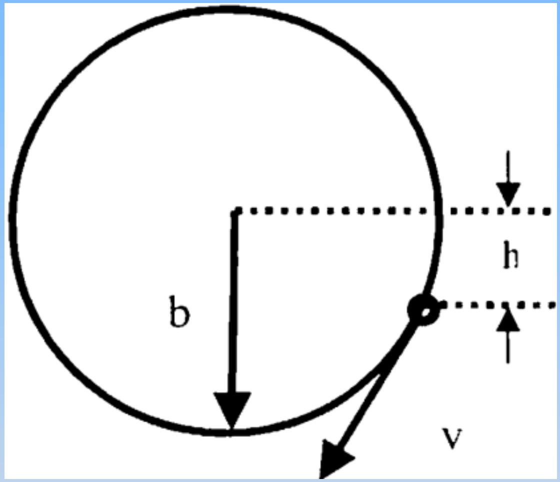
at the bottom of the loop, $h = -b$

so $\frac{1}{2}mv^2 = mgb$.

$$v = \sqrt{2gb}$$

$$F_r = -mg + R = \frac{mv^2}{b}$$

$$R = mg + \frac{mv^2}{b} = mg + 2mg = 3mg$$



General Motion of a Particle in Three Dimensions

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Spherical Pendulum

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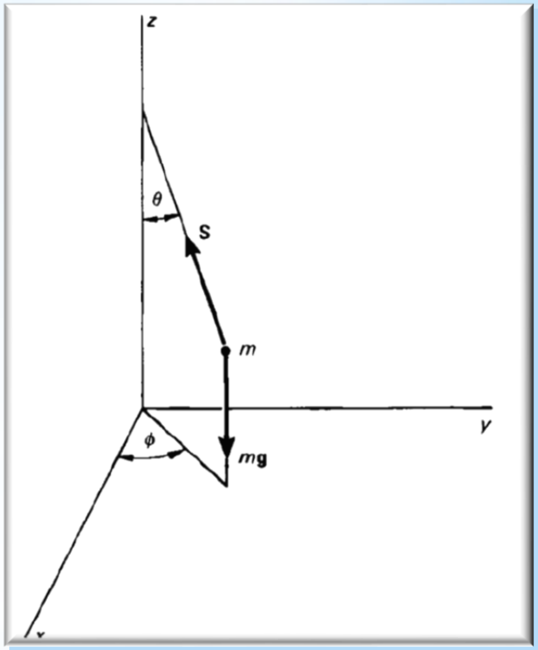
The spherical pendulum:

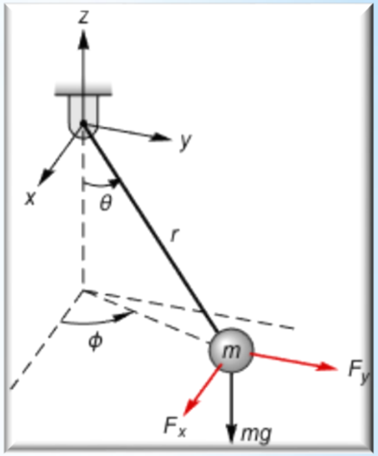
The case is illustrated by a heavy bob attached to a light inextensible rod or cord which is free to swing in any direction about a fixed point ,see in figure, this is called **spherical pendulum**.



There are two forces acting on the particle, namely the downward forces of gravity, and the tension **S** in the constraining rod or cord. The differential equation of motion is

$$m\ddot{\mathbf{r}} = m\mathbf{g} + \mathbf{S}$$

If we choose the z-axis to be vertical, the rectangular component of the equation of motion are as the follows:



$$m\ddot{x} = S_x$$
$$m\ddot{y} = S_y$$
$$m\ddot{z} = S_z - mg$$

An approximate solution is obtained for the case in which the displacement from the equilibrium position is very small. The magnitude of the tension is then very nearly constant and equal to mg, and we have $|x| \ll l$, $|y| \ll l$, $z = 0$. The x and y component of **S** are then given by the approximate relations

$$S_x \simeq -mg\,\frac{x}{l}$$

$$S_y \simeq -mg\,\frac{y}{l}$$

The x-y differential equation of motion then reduce to

$$\ddot{x} + \frac{g}{l}x = 0$$

$$\ddot{y} + \frac{g}{l}y = 0$$

The solution are:

$$x = A\cos(\omega t + \alpha)$$
$$y = B\cos(\omega t + \beta)$$

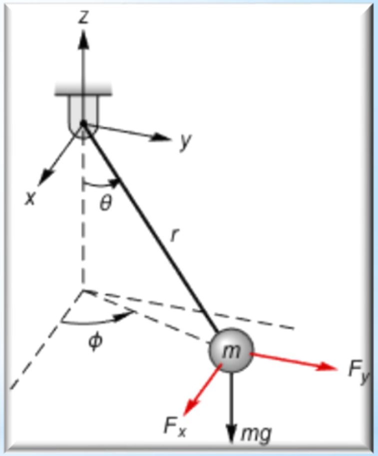
in which

$$\omega = \left(\frac{g}{l}\right)^{1/2}$$

Solution in spherical coordinate:

For more accurate treatment of the spherical pendulum we shall employ spherical coordinate. The tension S has only a radial component, but the weight *mg* has both a radial component *mg*cosθ and a transverse component –*mg*sinθ. Hence the differential equation of motion can be resolved in to spherical components as follow:

$$ma_r = F_r = mg\cos\theta - S$$
$$ma_\theta = F_\theta = -mg\sin\theta$$
$$ma_\varphi = F_\varphi = 0$$



since the constraint is that:

$$r = l = \text{constant}$$

and the acceleration in spherical components is:

$$\mathbf{a} = (\ddot{r} - r\dot{\phi}^2\sin^2\theta - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\mathbf{e}_\theta + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\mathbf{e}_\phi$$

we can ignore the radial component of the acceleration. The other two component reduce to:

$$a_\theta = l\ddot{\theta} - l\dot{\varphi}^2 \sin\theta \cos\theta$$
$$a_\varphi = l\ddot{\varphi}\sin\theta + 2l\dot{\varphi}\dot{\theta}\cos\theta$$

Thus, after transposing and performing cancellations, the differential equations in θ and ϕ become

$$\ddot{\theta} - \dot{\varphi}^2 \sin\theta\cos\theta + \frac{g}{l}\sin\theta = 0 \quad \cdots\cdots\cdots \quad (1)$$

$$\frac{1}{\sin\theta}\frac{d}{dt}(\dot{\varphi}\sin^2\theta) = 0 \quad \cdots\cdots\cdots\cdots \quad (2)$$

The second equation implies that the quantity in the parentheses is constant let call *h*. it is the angular momentum (per unit mass) about the vertical axis. The reason that it is constant stems from the absence of any moment of force about that axis. Then we can write:

$$\dot{\varphi} = \frac{h}{\sin^2\theta}$$

Upon inserting the above value of $\dot{\varphi}$ in to equ.(1), we obtain the separated equation in θ:

$$\ddot{\theta} - \dot{\varphi}^2 \sin\theta \cos\theta + \frac{g}{l} \sin\theta = 0$$

$$\ddot{\theta} + \frac{g}{l} \sin\theta - h^2 \frac{\cos\theta}{\sin^3\theta} = 0 \qquad \ldots\ldots\ldots (3)$$

Consider some special cases at this point. First, if the angle ϕ is constant then $\dot{\varphi}$ =0 and so h = 0, equ. (1) reduce to:

$$\ddot{\theta} + \frac{g}{l} \sin\theta = 0$$

Is just the differential equation of **simple pendulum**.

the second special case is that of the conical pendulum; $\theta = \theta_o$= constant. In this case $\dot{\theta} = 0$, $\ddot{\theta} = 0$ so equ.(3) reduce to:

$$\frac{g}{l} \sin\theta_0 - h^2 \frac{\cos\theta_0}{\sin^3\theta_0} = 0$$

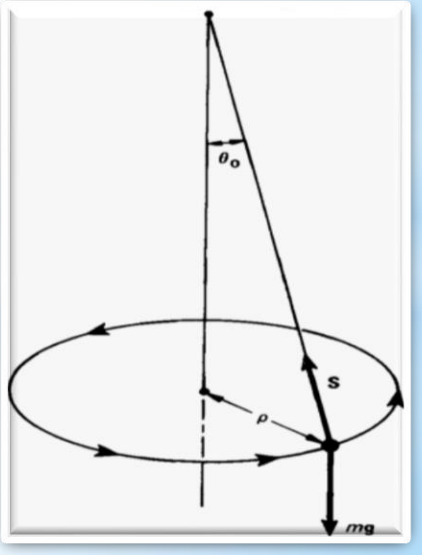
$$h^2 = \frac{g}{l} \sin^4\theta_0 \sec\theta_0 \qquad \ldots\ldots\ldots\ldots (4)$$

from the value of *h* given above we find:

$$\dot{\varphi}_0^2 = \frac{g}{l} \sec \theta_0 \quad \text{..............(5)}$$

As the condition for conical motion of the pendulum. The above equation can be obtained by considering the forces acting on the particle in its circular motion as shown in figure. The acceleration is constant in magnitude, and its directed toward the center of the circular path. Hence, upon taking horizontal and vertical component, we have:

$$S \sin \theta_0 = (ml \sin \theta_0)\dot{\varphi}_0^2$$
$$S \cos \theta_0 = mg$$



Which reduce to equation (5) elimination of **S**.

Let us now consider the case in which the motion is almost conical; the value of θ remains close to the value of θ_o. if we insert the expression for h^2 into the separated differential equation for θ, the result is

$$\ddot{\theta} + \frac{g}{l}\left(\sin\theta - \frac{\sin^4\theta_0}{\cos\theta_0}\frac{\cos\theta}{\sin^3\theta}\right) = 0$$

General Motion of a Particle in Three Dimensions

المادة الميكانيك التحليلي

الصف الثالث

د. علي عباس محمد صالح

المحاضرة الرابعة: حركة القذيفة في مجال تثاقلي بوجود مقاومة الهواء

- الحركة المقيدة للجسيم

*** Motion of a Projectile in a Uniform Gravitational Field**

*** Constrained motion of a particle**

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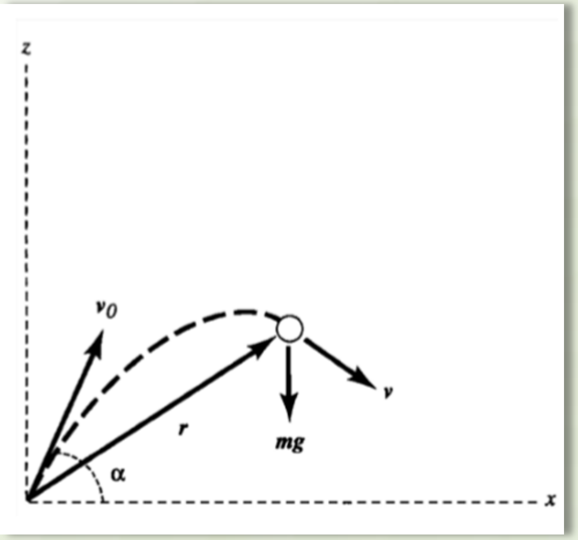
Motion of a Projectile in a Uniform Gravitational Field

B: Linear Air Resistance:

We now consider the motion of a projectile subject to the force of air resistance. In this case, the motion does not conserve total energy, which continually diminishes during the flight of the projectile . To simplify the resulting equation of motions, we take the constant of proportionality to be mγ where m is the mass of the projectile. The equation of motion is then:

$$\boldsymbol{F} = \boldsymbol{ma} = \boldsymbol{F} + \boldsymbol{G}$$

$$m\frac{d^2\mathbf{r}}{dt^2} = -m\gamma\mathbf{v} - \mathbf{k}mg \qquad \ldots\ldots\ldots\ldots(1)$$



Upon canceling m's, the equation simplifies to:

$$\frac{d^2\mathbf{r}}{dt^2} = -\gamma\mathbf{v} - \mathbf{k}g \qquad \ldots\ldots\ldots\ldots(2)$$

Before integrating, we write Equation (2) in component form

$$\ddot{x} = -\gamma \dot{x}$$
$$\ddot{y} = -\gamma \dot{y}$$
$$\ddot{z} = -\gamma \dot{z} - g$$

We see that the equations are separated; therefore, each can be solved individually. We can write down the solutions, noting that here γ= c1/m, c1 being constant. The results for the velocity components are:

$$\dot{x} = \dot{x}_0 e^{-\gamma t}$$
$$\dot{y} = \dot{y}_0 e^{-\gamma t}$$
$$\dot{z} = \dot{z}_0 e^{-\gamma t} - \frac{g}{\gamma}(1 - e^{-\gamma t})$$

We orient the coordinate system such that the x-axis lies along the projection of the initial velocity onto the xy horizontal plane. Then $\dot{y} = \dot{y}_o = 0$ and the motion is confined to the xz vertical plane. Integrating once more, we obtain the position coordinates

$$x = \frac{\dot{x}_0}{\gamma}(1 - e^{-\gamma t})$$

$$z = \left(\frac{\dot{z}_0}{\gamma} + \frac{g}{\gamma^2}\right)(1 - e^{-\gamma t}) - \frac{g}{\gamma} t$$

We have taken the initial position of the projectile to be zero, the origin of the coordinate system. This solution can be written vectorially as:

$$\mathbf{r} = \left(\frac{\mathbf{v}_0}{\gamma} + \frac{\mathbf{k} g}{\gamma^2}\right)(1 - e^{-\gamma t}) - \mathbf{k}\frac{gt}{\gamma}$$

Contrary to the case of zero air resistance the path of the projectile is not a parabola, but rather a curve that lies below the corresponding parabolic trajectory. Inspection of the x equation shows that, for large t, the value of x approaches the limiting value

$$x \to \frac{\dot{x}_0}{\gamma}$$

This means that the complete trajectory of the projectile, if it did not hit anything, would have a vertical asymptote as shown in Figure.

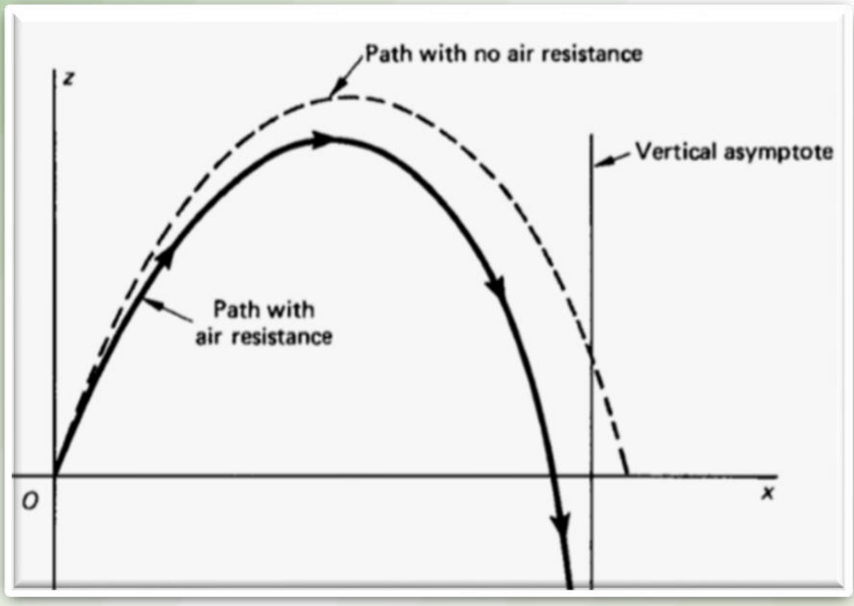


Figure: Comparison of the paths of a projectile with and without air resistance.

- **Constrained Motion of a Particle:**

When a moving particle is restricted geometrically in the sense that it must stay on a certain definite surface or curve, the motion is said to be constrained. A piece of ice sliding around a bowl and a bead sliding on a wire are examples of constrained motion

- **The Energy Equation for Smooth Constraints .**

The total force acting on a particle moving under constraint can be expressed as the vector sum of the net external force F and the force of constraint R. The latter force is the reaction of the constraining agent upon the particle. The equation of motion may, therefore, be written

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} + \mathbf{R}$$

If we take the dot product with the velocity v, we have

$$m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \mathbf{F}\cdot\mathbf{v} + \mathbf{R}\cdot\mathbf{v}$$

Now in the case of a smooth constraint the reaction **R** is normal to the surface or curve while the velocity **v** is tangent to the surface. Hence, **R** is perpendicular to **v**, and the dot product **R . v** vanishes. Equation above then reduces to:

$$\frac{d}{dt}\left(\frac{1}{2}m\mathbf{v}\cdot\mathbf{v}\right) = \mathbf{F}\cdot\mathbf{v}$$

If **F** is conservative we find that even though the particle is constrained to move along the surface or curve, its total remains constant, namely

$$\frac{1}{2}mv^2 + V(x,y,z) = \text{constant} = E$$

We have expected this to be the case for **frictionless constraints**

Example:

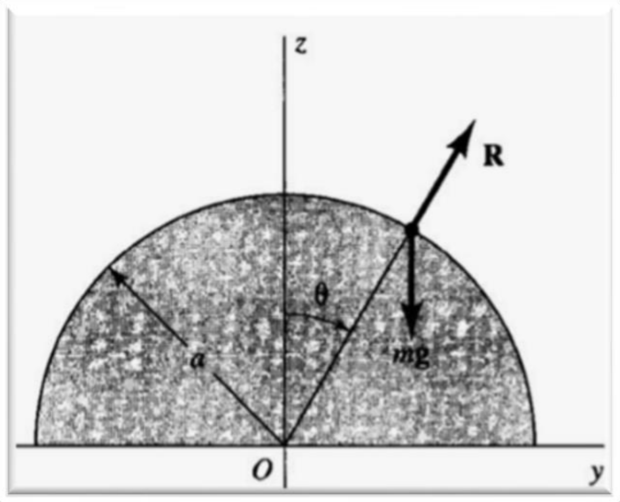
A particle is placed on top of a smooth sphere of radius a. If the particle is slightly disturbed, at what point will it leave the sphere?

Solution:

The forces acting on the particle are the downward force of gravity and the reaction R of the spherical surface. The equation of motion is:

$$m\frac{d\mathbf{v}}{dt} = m\mathbf{g} + \mathbf{R}$$

Let us choose coordinate axes as shown in Figure.



The potential energy is then mgz, and the energy equation reads:

$$\frac{1}{2}mv^2 + mgz = E$$

From the initial conditions (v = 0 for z = a) we have E = mga, so, as the particle slides down, its speed is given by the equation:

$$v^2 = 2g(a - z)$$

Now, if we take radial components of the equation of motion, we can write the force equation as:

$$-\frac{mv^2}{a} = -mg\cos\theta + R = -mg\frac{z}{a} + R$$

$$R = mg\frac{z}{a} - \frac{mv^2}{a} = mg\frac{z}{a} - \frac{m}{a}2g(a - z)$$

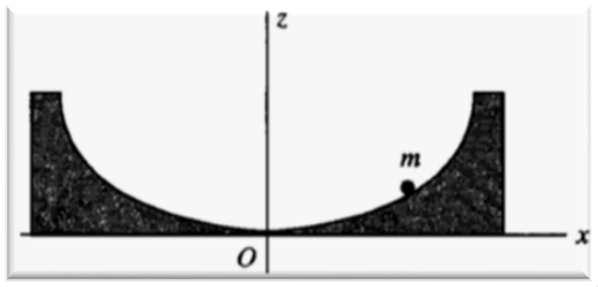
$$= \frac{mg}{a}(3z - 2a)$$

Thus, R vanishes when z =2/3a which point the particle leaves the sphere. This may be argued from the fact that the sign of R changes from positive to negative there.

Example:

Consider a particle sliding under gravity in a smooth cycloidal trough , as shown in figure, represented by the parametric equations:

$$x = A(2\phi + \sin 2\phi)$$
$$z = A(1 - \cos 2\phi)$$



Where ϕ is the parameter. **Find** the energy equation for the motion

Solution:

The energy equation for the motion, assuming no y-motion, is:

$$E = \frac{m}{2}v^2 + V(z) = \frac{m}{2}(\dot{x}^2 + \dot{z}^2) + mgz$$

Because $\dot{x} = 2A\dot{\phi}(1 + \cos 2\phi)$ and $\dot{z} = 2A\dot{\phi}\sin 2\phi$, we find the following expression for the energy in terms of ϕ:

$$E = 4mA^2\dot{\phi}^2(1 + \cos 2\phi) + mgA(1 - \cos 2\phi)$$

By use of the identities

$$1 + \cos 2\phi = 2\cos^2\phi \text{ and } 1 - \cos 2\phi = 2\sin^2\phi,$$

$$E = 8mA^2\dot{\phi}^2 \cos^2\phi + 2mgA\sin^2\phi$$

Let us introduce the variable s defined by s = 4A sinϕ. The energy equation can then be written:

$$E = \frac{m}{2}\dot{s}^2 + \frac{1}{2}\left(\frac{mg}{4A}\right)s^2$$

This is just the energy equation for harmonic motion in the single variable s. Thus, the particle undergoes periodic motion whose frequency is independent of the amplitude of oscillation ,unlike the simple pendulum for which the frequency depends on the amplitude. The periodic motion in the present case is said to be isochronous . (The linear harmonic oscillator under Hooke's law is, of course, isochronous.)

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Forces of separable type

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Forces of the Separable Type: Projectile Motion

The components of a force field involve the respective coordinates alone, that is,

$$\mathbf{F} = \mathbf{i}F_x(x) + \mathbf{j}F_y(y) + \mathbf{k}F_z(z)$$

Forces of this type are separable. In which the curl of such a force is identically zero. The field is conservative because each partial derivative is of the mixed type and vanishes identically, because the coordinates x, y, and z are independent variables. If the force is separable, then the component equations of motion are of the form

$$m\ddot{x} = f_x(x, \dot{x}, t)$$

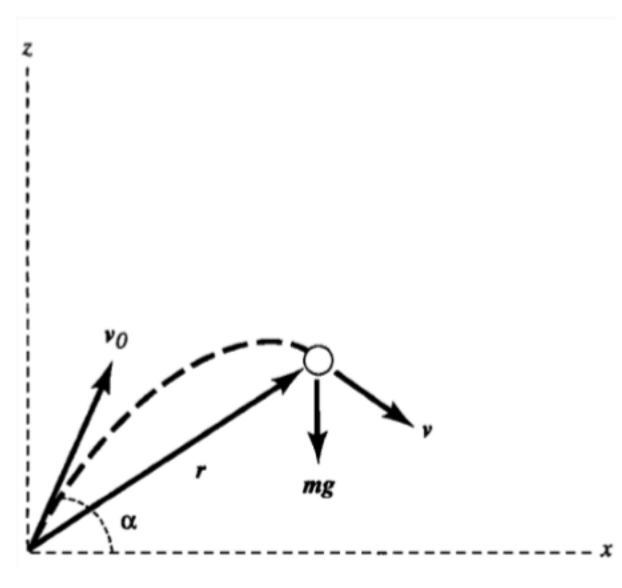
Motion of a Projectile in a Uniform Gravitational Field

A: **No Air Resistance:**

we first consider the case of a projectile moving with no air resistance. Only one force, gravity, acts on the projectile. Choosing the z-axis to be vertical, we have the following equation of motion:

$$m\frac{d^2\mathbf{r}}{dt^2} = -\mathbf{k}mg \qquad \ldots\ldots\ldots 1$$

Because there are no horizontally directed forces acting on the projectile, the motion occurs solely in the rz vertical plane. Thus, the position of the projectile at any time is: (see figure)



$$\mathbf{r} = \mathbf{i}x + \mathbf{k}z$$

The speed of the projectile can be calculated as a function of its height, z, using the energy equation:

$$\frac{1}{2}m(\dot{x}^2+\dot{z}^2)+mgz=\frac{1}{2}mv_0^2$$

or equivalently,

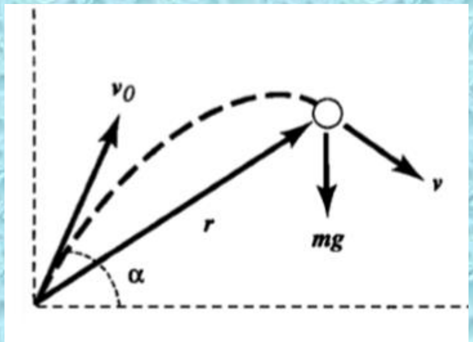
$$v^2=v_0^2-2gz$$

We can calculate the velocity of the projectile at any instant of time by integrating equation (1) to have:

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=-\mathbf{k}gt+\mathbf{v}_0 \qquad m\frac{d^2\mathbf{r}}{dt^2}=-\mathbf{k}mg$$

The constant of integration is the initial velocity vo. In terms of unit vectors, the velocity is:

$$\mathbf{v}=\mathbf{i}v_0\cos\alpha+\mathbf{k}(v_0\sin\alpha-gt)$$



Integrating once more yields the position vector:

$$\mathbf{r} = -\mathbf{k}\frac{1}{2}gt^2 + \mathbf{v}_0 t + \mathbf{r}_0$$

The constant of integration is the initial position of the projectile, r_o, which is equal to zero; therefore, in terms of unit vectors become:

$$\mathbf{r} = \mathbf{i}(v_0 \cos\alpha)t + \mathbf{k}\left((v_0 \sin\alpha)t - \frac{1}{2}gt^2\right)$$

In terms of components, the position of the projectile at any instant of time is:

$$x = \dot{x}_0 t = (v_0 \cos\alpha)t$$

$$y = \dot{y}_0 t \equiv 0 \qquad \ldots\ldots (2)$$

$$z = \dot{z}_0 t - \frac{1}{2}gt^2 = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$$

Where $\dot{x}_0 = v_0 \cos\alpha$, $\dot{y}_0 = 0$, and $\dot{z}_0 = v_0 \sin\alpha$ are the components of the initial velocity v_o.

We find z(x) by using the first of Equations (2) to solve for t as a function of x and then substitute the resulting expression in the third of Equations (2) to show that the path of the projectile is a **parabola.**

$$t = \frac{x}{v_0 \cos\alpha}$$

$$z = (\tan\alpha)x - \left(\frac{g}{2v_0^2 \cos^2\alpha}\right)x^2$$

$$x = \dot{x}_0 t = (v_0 \cos\alpha)t$$
$$y = \dot{y}_0 t \equiv 0$$
$$z = \dot{z}_0 t - \tfrac{1}{2}gt^2 = (v_0 \sin\alpha)t - \tfrac{1}{2}gt^2$$

We calculate several properties of projectile motion:

- First, we calculate the maximum height obtained by the projectile by using equation $v^2 = v_o^2 - 2gz$ and noting that at maximum height the vertical component of the velocity of the projectile is zero so that its velocity is in the horizontal direction and equal to the constant horizontal component, $v_o\cos\alpha$. Thus:

$$v_0^2 \cos^2\alpha = v_0^2 - 2gz_{max}$$

We solve this to obtain

$$z_{max} = \frac{v_0^2 \sin^2\alpha}{2g}$$

- **Second**, the time it takes to reach maximum height can be obtained from equation $\mathbf{v} = \mathbf{i}\, v_o \cos \propto\ + \mathbf{k}(v_o \sin \propto\ - gt)$ where we again make use of the fact that at maximum height, the vertical component of the velocity vanishes, so:

$$v_0 \sin\alpha - g t_{max} = 0 \quad \Rightarrow \quad t_{max} = \frac{v_0 \sin\alpha}{g}$$

- **Third**, we can obtain the total time of flight T of the projectile by setting z =0 in the equations (2), which yields:

$$T = \frac{2 v_0 \sin\alpha}{g}$$

This is twice the time it takes the projectile to reach maximum height. This indicates that the upward flight of the projectile to the apex of its trajectory is symmetrical to its downward flight away from it.

- **Fourth,** we calculate the range of the projectile by substituting the total time flight, T, into the first of equations (2), obtaining

$$R = x = \frac{v_0^2 \sin^2 2\alpha}{g}$$

$$x = \dot{x}_0 t = (v_0 \cos \alpha) t$$

$$T = \frac{2v_0 \sin \alpha}{g}$$

R has its maximum value

$$R_{max} = v_0^2/g \text{ at } \alpha = 45°$$