جامعة الموصل
كلية التربية للعلوم الصرفة
قسم الفيزياء

رياضيات متقدمة

Advanced Mathematics

المرحلة الثانية
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المحاضرة الأولى

المتسلسلات

The Series

Arithmetic Series:

The sum of the terms of an arithmetic sequence is called as arithmetic series.

$7, 17, 27, 37, 47, \ldots \Rightarrow$ is an Arithmetic sequence

$7 + 17 + 27 + 37 + 47 + \cdots \Rightarrow$ is Arithmetic series

The sum of n terms of an Arithmetic Series:

$$S_n = n\left[\frac{a_1 + a_n}{2}\right], a_n = a_1 + (n-1)d$$

Where:

a_1: *is the first term*

a_n: *is the n^{th} term*

n: *is the number of terms*

d: *is the common difference*

Example 1: Find the sum of the sequence $3, 9, 15, 21, \ldots$ to 10 terms.

Solution:

$S_n = \frac{n}{2}[2a_1 + (n-1)d], \qquad a_1 = 3$

$d = a_n - a_{n-1} = 9 - 3 = 6$

$S_{10} = \frac{10}{2}[2(3) + (10-1)6] \Rightarrow S_{10} = 5[6 + 54]$

$S_{10} = 5[60] = 300$

Example 2: Find the sum of the 60 terms of the following series

$9 + 14 + 19 + 24 + \cdots + 289 + 294 + 299 + 304$

Solution:

$\boldsymbol{d} = a_n - a_{n-1} = 24 - 19 = 5, \ \boldsymbol{n} = 60, \ \boldsymbol{a_1} = 9, \ \boldsymbol{a_{60}} = 304$

1. $S_n = \frac{n}{2}[2a_1 + (n-1)d] \Rightarrow S_{60} = \frac{60}{2}[2(9) + (60-1)(5)]$

   $S_{60} = 30[18 + 59(5)] \Rightarrow S_{60} = 30(313) = 9390$

2. $S_n = n\left[\frac{a_1 + a_n}{2}\right] \Rightarrow S_{60} = 60\left[\frac{9+304}{2}\right] \Rightarrow S_{60} = 30(313)$

   $S_{60} = 9390$

Geometric Series:

The geometric series is a series that has the form $\sum_{n=0}^{\infty} ar^n$, where (a) is a real constant and (r) is a real number.

$$\sum_{n=0}^{\infty} ar^n \quad = \quad S_n = a + ar + ar^2 + ar^3 + \cdots + ar^n$$

Example: find the sum of the series $\sum_{k=1}^{\infty} \frac{1}{2^k}$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots$$

$$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

Example: Write out the first four terms of the series $\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n$

$$\sum_{n=3}^{\infty} \left(\frac{2}{3}\right)^n = \underbrace{\frac{8}{27}}_{a} + \underbrace{\frac{16}{81}}_{ar} + \underbrace{\frac{32}{243}}_{ar^2} + \underbrace{\frac{64}{729}}_{ar^3} + \cdots$$

المحاضرة الثانية
الغايات
The Limit

Rules of limits: Useful rules for finding limits

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)}, where\ g(x) \neq 0$$

Example:

$$\lim_{x \to 1} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to 1}(x^4 + x^2 - 1)}{\lim_{x \to 1}(x^2 + 5)}$$

$$\frac{\lim_{x \to 1}(x^4) + \lim_{x \to 1}(x^2) - \lim_{x \to 1}(1)}{\lim_{x \to 1}(x^2) + \lim_{x \to 1}(5)} + \frac{1^4 + 1^2 - 1}{1^2 + 5} = \frac{1}{6}$$

- **Rule no.1**

1. If the degree of the numerators is lower than the degree of the denominator, the numerator be expressed as $n(x)$, and the denominator be expressed as $d(x)$, then:

$$\lim_{x \to \infty} \frac{n(x)}{d(x)} = 0$$

Example:

$$\lim_{x\to\infty}\frac{5x^3-2x^2+17x-13}{12-2x+x^4}=\lim_{x\to\infty}\frac{5x^3}{x^4}=\lim_{x\to\infty}\frac{5}{x}=0$$

Note: Any value divided by infinity equals 0

2. If the degree of the numerator is higher than the degree of denominator, then:

$$\lim_{x\to\infty}\frac{n(x)}{d(x)}=\infty\ or-\infty$$

Example:

$$\lim_{x\to\infty}\frac{3x^5+x^2-5}{6+x+7x^2}=\lim_{x\to\infty}\frac{3x^5}{7x^2}=\lim_{x\to\infty}\frac{3}{7}x^3=\infty$$

3. If the degree of the numerator is equal the degree of denominator, then is equal to the leading coefficient of $n(x)$ over the leading coefficient of $d(x)$.

Example:

$$\lim_{x\to\infty}\frac{3x^3-2x^2+x-4}{4+2x-5x^3}=\lim_{x\to\infty}\frac{3x^3}{-5x^3}=\lim_{x\Rightarrow\infty}-\frac{3}{5}=-\frac{3}{5}$$

- **Rule no.2**

L'Hôpital's Rule: It is a rule used in Limits, when the results of the Limits are not specified

If the $\lim_{x\to c}\frac{f(x)}{g(x)}$ results in one of the following forms:

$$\frac{0}{0}, \pm\frac{\infty}{\infty}, \infty \pm \infty, And\ \lim_{x\to c}\frac{f'(x)}{g'(x)}\ exits\ and\ g'(x) \neq 0, then:$$

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{\lim_{x\to c} f'(x)}{\lim_{x\to c} g'(x)}\ This\ is\ called\ L'Hôpital's\ Rule$$

Example 1: Find the $\lim_{x\to 0}\frac{e^x-1}{x}$

Solution:

$$\lim_{x\to 0}\frac{e^x - 1}{x} = \frac{\lim_{x\to 0}(e^x - 1)}{\lim_{x\to 0}(x)} = \frac{0}{0}$$

Using L'Hôpital's Rule

$$\lim_{x\to 0}\frac{e^x}{1} \Rightarrow \lim_{x\to 0}\frac{1}{1} = 1$$

Example 2: Find $\lim_{x\to 0}\frac{e^x-1-x-(x^2/2)}{x^3}$

Solution:

$\lim_{x\to 0}\frac{e^x-1-x-(x^2/2)}{x^3} = \frac{0}{0}$, Using L'Hôpital's Rule

$\lim_{x\to 0}\frac{e^x-1-x}{3x^2} = \frac{0}{0}$, Using L'Hôpital's Rule again

$\lim_{x\to 0}\frac{e^x - 1}{6x} = \frac{0}{0}$, Using L'Hôpital's Rule $\lim_{x\to 0}\frac{e^x}{6} = \frac{e^0}{6} = \frac{1}{6}$

المحاضرة الثالثة

تقارب وتباعد المتسلسلات

Convergence and Divergence of Series

1. Convergence and Divergence of Arithmetic Series:

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$S_n = a_1 + a_2 + a_3 + a_4 + \cdots + a_n$$

$$S_n = \frac{n}{2}[a_1 + a_n] \ or \ S_n = \frac{n}{2}[2a_1 + (n-1)d]$$

$$S_n = \begin{cases} Convergent, where \ \lim_{n\to\infty} S_n = L \\ Divergent, where \ \lim_{n\to\infty} S_n = \pm\infty \end{cases}$$

a_1: *The first term.*

S_n: *The sum of terms from 1 to n.*

L: *Specific number.*

Example 1: Test the following series for convergence or divergence $\sum_{n=1}^{\infty}(2n-1)$

Solution:

$$\sum_{n=1}^{\infty}(2n-1) = 1 + 3 + 5 + 7 + \cdots + (2n-1) + \cdots$$

$$d = a_n - a_{n-1} = 5 - 3 = 7 - 5 = 2 = d, and \ a_1 = 1$$

$$S_n = \frac{n}{2}[a_1 + a_n] \Rightarrow S_n = \frac{n}{2}[1 + (2n-1)] \Rightarrow S_n = \frac{n}{2}[1 + 2n - 1]$$

$$S_n = \frac{n}{2}[2n] \Rightarrow S_n = n^2 \Rightarrow \lim_{n\to\infty} S_n = \lim_{n\to\infty} n^2 = \infty$$

$$\sum_{n=1}^{\infty}(2n-1) \ is \ divergent.$$

2. Convergence and Divergence of Geometric Series:

$$\sum_{n=1}^{\infty} ar^n = ar + ar^2 + ar^3 + \cdots + ar^n$$

$$r = \begin{cases} |r| < 1, then\ the\ series\ is\ convergent, and\ S_n = \dfrac{a}{1-r} \\ |r| \geq 1, then\ the\ series\ is\ divergent, and\ It\ has\ no\ sum \end{cases}$$

a: *the first term.*

r: *The result of dividing each term by the previous term.*

S_n: *The sum of terms from* 1 *to* n.

Example 1: Test the following series for convergence or divergence $\sum_{i=1}^{\infty} \frac{1}{2^i}$,

if convergent then find the sum of this series.

Solution:

$$\sum_{i=1}^{\infty} \frac{1}{2^i} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \cdots + \frac{1}{2^n} + \cdots$$

$$r = \frac{a_n}{a_{n-1}} = \frac{\frac{1}{2^2}}{\frac{1}{2}} = \frac{\frac{1}{2^3}}{\frac{1}{2^2}} = \frac{1}{2}, |r| = \frac{1}{2} < 1, then\ it\ is\ convergent.$$

$$S_n = \frac{a}{1-r} \Rightarrow S_n = \frac{\frac{1}{2}}{1-\frac{1}{2}} \Rightarrow S_n = \frac{\frac{1}{2}}{\frac{1}{2}} \Rightarrow S_n = \frac{1}{2} * 2 = 1$$

Therefore the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$ *is convergent, and the sum is* 1.

3. Convergence and Divergence of P-series:

$$\sum_{n=1}^{\infty} \frac{1}{n^p}, p: is\ constant\ number.$$

$$p = \begin{cases} p > 1, then\ the\ series\ is\ convergent \\ 0 < p \leq 1, then\ the\ series\ is\ divergent \end{cases}$$

Example 1: Show that $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is convergent?

Solution:

$This\ series\ is\ p - series, where\ p = 2 > 1$

$threrfore\ the\ series$ $\sum_{r=1}^{\infty} \frac{1}{r^2}$ is convergent.

المحاضرة الرابعة

متسلسلة القوى ومتسلسلة تايلر

Power series and Taylor series

Centered of Power Series:

This series $\sum_{n=0}^{\infty} a_n(x-c)^n$ *is centered at*(c), *where* $c = a = x$

Example: where the following series are centered?

1. $\sum_{n=0}^{\infty} x^n$ ***centered at*** $c = 0$

2. $\sum_{n=0}^{\infty} \left(-\frac{1}{3}\right)^n (x-3)^n$ ***centered at*** $c = 3$

Radius and Interval of convergence of the power series:

for any power series there are only 3 possibilities for the value of (x) for which the series converges:

$$1. |x| * \infty = \begin{cases} \text{convergent only for } x = 0 \\ \text{Interval} = 0 \\ \text{Radius} = 0 \end{cases}$$

$$2. |x| * 0 = \begin{cases} \text{convergent for all value of } x \\ \text{interval} = (-\infty, +\infty) \\ \text{Radius} = \infty \end{cases}$$

3. $|x| * number$

Example 5: Determine the radius and interval of convergence for $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$

Solution:

Using Ratio Test: $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} \Rightarrow \lim_{n\to\infty} \left| \frac{(x+1)^{n+1}}{(n+1)!} * \frac{n!}{(x+1)^n} \right|$

$$= \lim_{n\to\infty} \left| \frac{x+1}{n+1} \right| = |x+1| \lim_{n\to\infty} \frac{1}{n+1} = |x+1| * 0 = 0 < 1$$

The series converges for all value of x.

Interval $= (-\infty, \infty)$

Radius $= \infty$

Example 6: Find the radius and interval of convergence of the series $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n^2}$

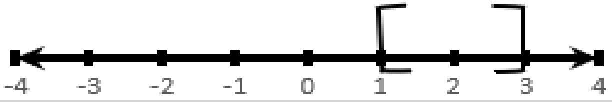
Solution:

Using Ratio Test: $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \left| \frac{\frac{(x-2)^{n+1}}{(n+1)^2}}{\frac{(x-2)^n}{n^2}} \right| = \lim_{n\to\infty} \frac{(x-2)^{n+1}}{(n+1)^2} * \frac{n^2}{(x-2)^n}$

$$= |x-2| \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^2 = |x-2| * 1 = |x-2|$$

The series converges when $|x-2| < 1 \Rightarrow -1 < x-2 < 1 \Rightarrow 1 < x < 3$

The Interval $= (1,3)$



Taylor Series

A Taylor series is an infinite sum that represents a particular function, The Taylor series for a function $f(x)$, centered at $x = a$, is the infinite series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Note: $0! = 1$.

Example 1: Find the first five terms of the Taylor series for: $f(x) = \sin x$, **centered at $a = 0$?**

Solution:

$f(x) = \sin x \Rightarrow f(0) = \sin 0 = 0$

$f'(x) = \cos x \Rightarrow f'(0) = \cos 0 = 1$

$f''(x) = -\sin x \Rightarrow f''(0) = -\sin 0 = 0$

$f'''(x) = -\cos x \Rightarrow f'''(0) = -\cos 0 = -1$

$f''''(x) = \sin x \Rightarrow f''''(0) = \sin 0 = 0$

$$f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(0) + f^{(1)}(0)(x-0)^1 + \frac{f^{(2)}(0)}{2}(x-0)^2 + \frac{f^{(3)}(0)}{3!}(x-0)^3 + \cdots$$

The Taylor series for $f(x) = \sin x$, centered at $(a = 0)$ is:

$$0 + x + 0 - \frac{x^3}{3!} + 0 + \frac{x^5}{5!} + 0 - \frac{x^7}{7!} + 0 + \frac{x^9}{9!} - \cdots$$

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots$$

المحاضرة الخامسة

المعادلات التفاضلية الجزئية

Partial Differential Equation

- **First Order Partial Derivatives**:

$\frac{\partial z}{\partial x}$ is read as partial derivative of z with respect to x

and means differentiate with respect to x holding y constant.

$\frac{\partial z}{\partial y}$ is read as partial derivative of z with respect to y

and means differentiate with respect to y holding x constant.

Another common notation is the subscript notation:

Z_x means $\frac{\partial z}{\partial x}$ and Z_y means $\frac{\partial z}{\partial y}$

Example 1: Calculate $\frac{\partial z}{\partial x}$ *and* $\frac{\partial z}{\partial y}$ *when* $\boldsymbol{z = x^2 + 3xy + y - 1}$

Solution:

$$\frac{\partial z}{\partial x} = 2x + 3y$$

$$\frac{\partial z}{\partial y} = 3x + 1$$

- **Second Order Partial Derivatives:**

We express the second partial derivative in the form:

$$\frac{\partial^2 z}{\partial x^2} = z_{xx}, \qquad \frac{\partial^2 z}{\partial y^2} = z_{yy}$$

$$\frac{\partial^2 z}{\partial x \partial y} = z_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right), \qquad \frac{\partial^2 z}{\partial y \partial x} = z_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$$

$\frac{\partial^2 z}{\partial x^2}$ means the second derivative with respect to x holding y constant.

$\frac{\partial^2 z}{\partial y^2}$ means the second derivative with respect to y holding x constant.

$\frac{\partial^2 z}{\partial x \partial y}$ means differentiate first with respect to x and then with respect to y.

$\frac{\partial^2 z}{\partial y \partial x}$ means differentiate first with respect to y and then with respect to x.

Example 1: Find all second order partial derivatives of the following function $\boldsymbol{f(x.y) = x^2y^2 + y^2 + 2yx^3}$

Solution:

$$\frac{\partial f}{\partial x} = 2xy^2 + 6x^2y$$

$$\frac{\partial f}{\partial y} = 2x^2y + 2y + 2x^3$$

$$\frac{\partial^2 f}{\partial x^2} = 2y^2 + 12xy$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^2 + 2$$

$$\frac{\partial^2 f}{\partial y \partial x} = 4xy + 6x^2$$

$$\frac{\partial^2 z}{\partial x \partial y} = 4xy + 6x^2$$