

Lecturer :Asst. Prof. Dr. Hijran Abdulilah Al-Salihi

Subject :Mathematical Logic

Lecture Title :Symbolic Languages: Analytical Studies of G.E. Moore, Gottlob Frege, and David Hilbert

Academic Year2026-2025 :

Symbolic Languages

Section One: George Edward Moore's Symbolic Language

Introduction to George Moore

George Moore was an English philosopher (1873–1958) who played a major role in founding analytic philosophy. His logical philosophy was renowned for its attention to linguistic precision and analysis, as well as its quest to establish a rigorous scientific philosophy. His ideas helped clarify and resolve many philosophical dilemmas, especially in the field of moral philosophy, urging philosophers to focus on analyzing language with greater accuracy. Among his most prominent works are *Principia Ethica* ,*Philosophical Studies* ,and *Some Main Problems of Philosophy*.

George Moore's Contributions to Logical Studies

- **Developing the Theory of Meaning :**Moore introduced a new approach to meaning known as the Natural Theory of Meaning, which connects the meaning of words to the real world.
- **Studying Mathematical Logic :**Moore contributed to the advancement of mathematical logic, particularly in set theory.
- **Critique of Idealism :**Moore attacked idealistic philosophy, specifically Hegelian philosophy.
- **Defense of Realism :**Moore rejected idealism and agnosticism, defending the existence of a real world independent of the human mind.
- **Emphasizing the Importance of Experience :**Moore maintained that genuine knowledge originates from sensory experience, and any knowledge not grounded in experience is unreliable.

- **Distinguishing Between Analytic and Synthetic Truths :**Moore divided truths into two types: analytic truths, which are true by virtue of their definitions, and synthetic truths, which relate to the real world and require empirical evidence for validation.

Characteristics and Features of Moore's Symbolic Language

- **Precision :**Moore aimed to make symbolic language more precise than natural language by defining terms clearly and establishing strict rules for their usage.
- **Clarity :**He sought to enhance clarity by using specific symbols for each term, thereby avoiding ambiguous or multi-meaning words.
- **Simplicity :**Moore worked to keep symbolic language simpler than natural language by using a minimal number of symbols and rules while avoiding unnecessary complexities.
- **Logical Rigor :**Moore ensured that symbolic language was logically sound by verifying that all rules and laws governing its use remained consistent with the laws of logic.
- **Applicability :**He endeavored to make symbolic language applicable across various fields of philosophy and science by designing it to describe any subject or phenomenon accurately and clearly.

George Moore's Technical Inputs to Symbolic Language

- He helped develop a precise and clear symbolic language, making it a more effective tool in philosophical and scientific inquiry.
- He raised awareness about the vital importance of symbolic language within philosophy and science.
- His works inspired numerous other philosophers to adopt symbolic language in their own research.

George Moore's Logical Theory of Natural Meaning

Moore's theory of meaning is a realistic theory that links word meanings to the objective world. He rejected the notion that meaning consists

merely of ideas or emotions existing within the human mind. Instead, he asserted that meaning is an inherent property of things in the real world. This theory is built on several core principles:

- **Meaning is Objective** :Moore believed that meaning is not mental, but rather an objective property of things in the real world.
- **Meaning is Unanalyzable** :Moore posited that meaning is a primitive property that cannot be broken down into simpler elements. For instance, the meaning of the word "tree" cannot be fully analyzed merely by listing traits like "having leaves" and "having roots".
- **Meaning is Linked to Use** :Moore maintained that the meaning of words depends on how they are utilized within language.
- **Meaning is Bound to Sensory Experience** :Moore believed that we acquire the meaning of words through our physical interactions and experiences with the real world (e.g., learning the word "tree" by seeing and touching trees).

Advantages and Disadvantages of Moore's Theory of Meaning

- **Advantages:**
 - It helps clarify the nature of meaning and explains how it originates.
 - It assists in resolving certain philosophical dilemmas, such as the problem of the external world.
 - It enhances communication among people by clarifying the exact meanings of words.
- **Disadvantages:**
 - **Application Difficulties** :Some find it difficult to apply the natural theory of meaning to certain abstract words.
 - **Neglect of the Linguistic Aspect** :The theory focuses heavily on the factual, real-world side of meaning while neglecting structural linguistic mechanics.

- **Neglect of Context** :It does not sufficiently account for the varying contexts in which words are used.
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Section Two: Gottlob Frege's Symbolic Language

Introduction to Gottlob Frege

Gottlob Frege (1848–1925) is considered one of the most important pioneers of modern logic and analytic philosophy. Frege sought to develop a rigorous symbolic language capable of expressing thoughts and concepts with absolute precision and clarity, moving far away from the ambiguity of natural language.

Objectives of Frege's Symbolic Language

- **Precision** :It aims to bypass the ambiguity of natural language by utilizing symbols to represent concepts precisely, preventing misunderstandings and multiple conflicting interpretations.
- **Clarity** :It helps analyze and clarify complex ideas by breaking them down into their fundamental components and presenting them systematically.
- **Rigor** :It enables the construction of strict logical systems used to verify the validity of arguments and proofs.

Characteristics of Frege's Symbolic Language

- **Compositionality** :The language consists of a set of symbols combined according to specific rules to form meaningful expressions.
- **Symbolism** :Symbols are explicitly used to represent concepts and ideas, dedicating a specific symbol to each distinct concept.
- **Exactness** :The meanings of symbols are defined with absolute precision to avoid any ambiguity or confusion.
- **Formality** :It expresses thoughts abstractly, without relying on specific examples or localized cases.

Frege's Major Logical Achievements

- **The Symbolic Logic System** :Frege developed a symbolic logical system known as "Predicate Logic," used to express propositions and the relationships between concepts.
- **Distinction Between Sense and Reference) *Sinn und Bedeutung*** :(Frege famously distinguished between the sense of a symbol and its reference. The sense refers to the mode of presentation or meaning that the symbol carries, while the reference points directly to the actual object or concept that the symbol denotes.
- **Impact on Philosophy and Mathematics** :Frege's work profoundly influenced modern logic, analytic philosophy, and the foundations of set theory.

Limitations of Gottlob Frege's Symbolic Language

- **Difficulty** :Symbolic language can be difficult to learn and comprehend, particularly for individuals lacking a solid background in logic or mathematics.
- **Lack of Flexibility** :It is not well-suited for expressing all forms of human thought, especially those rooted in deep emotions or subjective personal experiences.
- **Dependence on Natural Language** :Despite Frege's quest to build an independent symbolic language, it cannot be entirely detached from natural language, as natural language remains necessary to explain the meanings of symbols and define grammatical rules.

Conclusion

Frege's symbolic language stands as a monumental milestone in the history of logic and philosophy. It enabled the precise analysis of complex thoughts and drove the development of modern logic and set theory. Nevertheless, it carries inherent drawbacks, including its steep learning curve and lack of expressive flexibility for non-formal concepts.

Section Three: David Hilbert's Symbolic Language

David Hilbert's Philosophy of Mathematics

David Hilbert was a celebrated German mathematician who lived between 1862 and 1943. He was widely renowned for his groundbreaking work in algebra, geometry, and mathematical analysis. However, Hilbert was not merely a mathematician; he was also a highly influential philosopher of science and mathematics.

Core Ideas in Hilbert's Philosophy of Mathematics

1. Hilbert's Program:

- In 1922, Hilbert introduced an ambitious research program aimed at placing the foundations of mathematics upon a completely secure and rigorous logical basis.
- The program sought to prove all mathematical truths from basic logical axioms using strict rules of inference.
- The program faced major structural challenges, most notably: the difficulty of finding a complete set of axioms, proving the absolute consistency of the axiomatic system, and proving that the system could encompass all domains of mathematics. Despite these challenges, Hilbert's program heavily shaped 20th-century mathematics and philosophy.

2. Intuitionism (Contextual View):

- Hilbert defended a view that recognized the role of human intuition and reasoning in mathematics. He argued that mathematics is not merely a formal logical game, but also a creative human activity involving intuition and deep understanding. He viewed intuition as a vital source of mathematical knowledge that plays an essential role in discovering new mathematical truths.

3. The Formalist School:

- Hilbert was a principal founder of Formalism in philosophy, which emphasizes the absolute importance of form and structure in language and thought. He argued that mathematics is a formal system that can be fully understood

by analyzing its structural syntax without any necessary reference to external semantic content. Hilbert's formalist ideas directly spurred the development of modern proof theory.

4. Epistemology:

- Hilbert made significant contributions to the theory of knowledge, particularly regarding the concepts of verification, certainty, and mathematical truth. He argued that mathematical knowledge is objective and anchored to strict logical foundations, helping clarify the nature of scientific knowledge as a whole.

Philosophy of Science

Hilbert maintained a keen interest in the philosophy of science, offering crucial insights into the nature of scientific evolution. He argued that science is an ongoing, continuous quest for truth, and that scientific knowledge shifts dynamically with new empirical and theoretical discoveries.

Summary

David Hilbert left a monumental legacy on 20th-century philosophy and mathematics. His foundational ideas regarding mathematics, formalism, intuition, epistemology, and the philosophy of science remain central subjects of debate and research among modern philosophers and mathematicians today.

Attributes and Characteristics of David Hilbert's Symbolic Language

David Hilbert viewed symbolic language theory as a vital tool for achieving a superior understanding of mathematics. His system is characterized by the following features:

1. **Formalism** :The symbolic language is a purely formal system composed of tokens and strict rules of combination. Symbols do not inherently point to specific real meanings; instead, they represent highly abstract concepts. The structural rules are used to build formulas and expressions with strict syntactical meaning.

2. **Precision** :Symbolic language is far more precise than natural language, systematically eliminating ambiguity, vagueness, and conflicting interpretations from mathematical expressions.
3. **Brevity** :It is highly concise, allowing complex, multi-layered ideas to be expressed using a minimal number of symbols, which facilitates faster information processing.
4. **Universality** :It is a global language independent of any natural tongue. Anyone can understand symbolic language regardless of their native language, enabling seamless communication among mathematicians worldwide.
5. **Expressive Power** :The language possesses high expressive capacity, capable of articulating any mathematical idea and facilitating the formulation of entirely new mathematical theories.
6. **Independence** :It remains independent of the actual semantic content of the thoughts it expresses, focusing entirely on structural form rather than subjective meaning. This allows mathematical concepts to be analyzed objectively.
7. **Logical Alignment** :It is intrinsically bound to logic. The rules of symbolic language are deployed to construct formal logical formulas, allowing the absolute validity of mathematical theorems to be structurally proven.
8. **Impact** :It has exerted an immense influence on the evolution of mathematics, unifying various mathematical fields, maximizing precision, and becoming the standard operating language across all mathematical disciplines today.

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Lecturer :Asst. Prof. Dr. Hijran Abdulilah Al-Salihi

Subject :Mathematical Logic

Lecture Title :Advanced Logical Systems, Predicate Logic, and the Foundations of Set Theory

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Chapter Four: Logical Systems, Propositional Functions, Quantifiers, and the Method of Proof

Introduction

The transition from propositional logic to predicate logic represents an advanced stage in the evolution of mathematical logic. Attention is no longer confined to studying propositions as isolated units with fixed truth values; instead, the focus shifts toward analyzing the internal structure of the proposition, uncovering its constituent elements, and examining the relations that bind its parts. This development has led to the emergence of logical systems far more capable of expressing complex relations that simple formulas in propositional logic fail to capture.

A logical system refers to a structured set of symbols, definitions, and rules used to construct logical expressions and derive conclusions precisely and consistently. A logical system is not merely a collection of characters; rather, it is an integrated intellectual framework aimed at organizing thought and inference processes according to specific principles. Its importance lies in providing a formal language that permits the analysis of propositions far from the ambiguity that characterizes natural language.

The development of logical systems brought forth the concept of propositional functions, which is one of the most critical concepts in modern mathematical logic. A standard proposition possesses a specific truth value, whereas a propositional function is a symbolic expression containing one or more variables. It cannot be judged as true or false until these variables are replaced by specific values or individuals. For instance, if we say "x is human," this expression is not a complete proposition as long as the symbol (x) remains unspecified; however, it

transforms into a proposition with a definitive truth value when we replace the variable with a specific name, such as "Ali is human".

Consequently, it became possible to analyze the internal content of propositions with greater accuracy. The proposition is no longer treated as a closed unit, but rather as a structure consisting of a subject, a predicate, and the relation linking them. This analysis has allowed for uncovering the deep logical structures underlying various linguistic expressions, and has enabled logicians to study general relations and shared characteristics among individuals and things.

Propositional functions are intimately linked to the concept of variables. A variable is a symbol that can refer to an unspecified number of individuals or objects, serving as a fundamental tool for expressing generality and abstraction within mathematical logic. The use of variables has expanded logic's capacity to express general laws and universal judgments that cannot be represented through single, isolated propositions alone.

To define the scope of variables and employ them accurately, logical quantifiers were introduced as one of the most significant innovations in predicate logic. The universal quantifier expresses that a judgment encompasses all individuals within a domain, whereas the existential (particular) quantifier denotes the existence of at least one individual for whom the specified attribute or relation holds true. Quantifiers have contributed to transforming general and particular judgments into clear symbolic formulas that can be subjected to logical analysis and mathematical proof.

The importance of quantifiers stems from their ability to express universal and existential propositions precisely. When we say "All humans are mortal," we apply a universal judgment that includes every member of the human class. Conversely, when we say "Some students are industrious," we express the existence of specific individuals to whom the attribute applies, without extending the judgment to all members. This distinction has enabled logicians to construct more advanced systems to study the logical relations between different judgments.

Through the study of quantifiers, a clear distinction emerged between free variables and bound variables. A free variable is one that appears in a

propositional function without being subject to a logical quantifier that restricts its scope. On the other hand, a variable becomes bound when it falls within the scope of a universal or existential quantifier. This distinction is vital for understanding the logical structure of compound propositions and for determining the exact meaning of symbolic expressions.

The relationship between propositions and propositional functions represents a key topic in mathematical logic. A proposition can be viewed as a special case of a propositional function where the variables have been assigned specific values. Meanwhile, a propositional function represents the general formula that can generate a large number of distinct propositions depending on the values assigned to its variables. Hence, propositional functions have become an effective tool for expressing general laws, scientific theories, and mathematical relations.

This evolution has led to building an integrated network of theorems and rules of inference that allow deriving conclusions from premises according to strict logical foundations. A theorem is a proposition whose truth is proven based on a set of recognized definitions, axioms, and inference rules within the logical system. Theorems derive their importance from being necessary conclusions resulting from primary principles that are assumed to be free of error or contradiction.

The process of proof relies on a series of organized logical steps that begin with given premises and terminate at a necessary conclusion. A proof must be consistent, free of fallacies, and grounded upon valid rules of inference. Through these rules, one can move from particular propositions to general judgments or vice versa, and uncover hidden relationships linking different concepts.

The application of systems of definitions has advanced the deductive method within mathematical logic, making it possible to construct concepts precisely and determine their core properties before utilizing them in reasoning. A logical definition is not merely a linguistic explanation of a term; it is a formal tool that determines the domain of a concept, its boundaries, and its relations to other concepts. Thus, definitions have become an essential part of the logical architecture of any scientific or philosophical system.

The significance of these concepts is not confined to logic alone; it extends to mathematics, computer science, artificial intelligence, linguistics, and contemporary analytic philosophy. Modern computer programming relies heavily on variables, functions, and logical quantifiers, while artificial intelligence systems depend on rules of inference and proof to process information and make decisions.

Thus, it is clear that studying logical systems, propositional functions, quantifiers, and theorems represents an advanced stage of mathematical logic. It exposes the internal structure of human thought and provides precise tools to analyze the relations and inferences upon which various sciences are built. Through these tools, mathematical logic has managed to surpass the limits of traditional propositional logic, moving to a deeper and more comprehensive level in the study of thought, language, and knowledge.

Chapter Five: Set Theory and Operations Between Sets

Introduction

Set theory is one of the most vital foundations upon which the contemporary structure of mathematics and mathematical logic rests. Indeed, many researchers consider it the common language through which diverse mathematical and logical concepts can be expressed. This theory has occupied a central position since the late nineteenth century, when mathematicians and logicians sought more accurate and rigorous foundations to organize mathematical knowledge. They found in the concept of a set a powerful tool capable of unifying different concepts and relations within a single logical framework.

The importance of set theory is not limited to mathematics alone; it extends directly into logic, philosophy, computer science, artificial intelligence, and linguistics because it provides a precise means to express relations between objects, classify them, and study their shared characteristics. Consequently, set theory has become an essential part of studying modern mathematical logic, as it aids in understanding the nature of classification, abstraction, and the logical relations that connect concepts and individuals.

Origin and Development of Set Theory

The emergence of set theory is historically tied to the name of the German mathematician Georg Cantor, who is considered the actual founder of this branch of knowledge. His research initiated from an attempt to study infinite sets and distinguish between their different types—an endeavor that yielded results that triggered a profound shift in contemporary mathematical thought.

At their inception, Cantor's ideas faced substantial criticism and opposition, because the concept of infinity raised complex philosophical and mathematical paradoxes. However, subsequent developments proved the immense value of this theory and its role in establishing modern mathematics on clearer and more precise foundations.

With the advancement of mathematical logic, set theory became an indispensable tool for analyzing mathematical and logical structures, making it possible to express numbers, relations, functions, and various mathematical systems within a unified framework.

The Concept of a Set

A set is defined as a collection of well-defined objects or elements, such that it is possible to determine clearly whether any given element belongs or does not belong to it. The objects that make up the set are called its elements.

The concept of a set is characterized by a high degree of abstraction; a set can consist of material objects, mental concepts, numbers, symbols, individuals, or even other sets. When we speak of the set of students in a classroom, the set of even numbers, or the set of Muslim philosophers, we refer in each case to a collection of elements united by a specific description or property.

The clarity of the criterion of membership is the fundamental condition for the existence of a set, since ambiguity in specifying the elements leads to a loss of the required logical precision.

Methods of Defining Sets

A set can be defined in more than one way, depending on the nature of its constituent elements:

- **The Roster (Listing) Method** :This first method consists of listing the elements of the set directly when their number is finite and known.
- **The Set-Builder (Description) Method** :This second method relies on specifying the shared property that characterizes the elements of the set, without the need to list them all individually.

The importance of the description method becomes particularly evident when dealing with large or infinite sets whose elements are impossible to count completely. These methods reveal the abstract nature of sets, showing that the true value of a set lies not in how it is written, but in the precise determination of its elements and relations.

Classification of Sets

The expansion of set theory led to several classifications that help understand their nature and the relations existing between them:

- **By Number of Elements** :A distinction can be made between finite sets and infinite sets.
- **By Nature of Elements** :A distinction can be made between numerical sets, symbolic sets, and conceptual sets.

Sets can also be classified according to the relations of containment and membership, which has given rise to the concepts of subsets, universal sets, and other fundamental notions. Classification is a necessary step in logical analysis because it helps organize knowledge and uncover structural relationships between different concepts.

Types of Sets

Among the most critical types of sets addressed in mathematical logic and modern mathematics is the **Empty Set** ,which is the set that contains no elements at all. Despite being devoid of elements, it plays a structural role in building mathematical and logical theories.

There is also the **Universal Set**, which contains all possible elements within a specific field of discourse or investigation, serving as a baseline reference when studying relations and operations between sets.

Other important types include **Equal Sets**, which possess exactly the same elements, and **Equivalent Sets**, which have the same number of elements (cardinality) even if the nature of those elements differs.

Additionally, the concept of a **Subset** stands out, which is based on the containment of one set inside a larger one. This concept is one of the most widely utilized tools in mathematical and logical proofs.

Relations Between Sets

The study of sets is built upon a group of core relations that determine how sets connect with one another:

- **Membership** :($\in$) This relation links an individual element to a set.
- **Containment** :($\supseteq$) This relation links one set to another set.

The clear distinction between these two relations has eliminated much of the historical confusion in mathematical and logical thinking. Other relations pertain to equality, equivalence, overlapping, and disjointedness—all representing different forms of association between sets. The importance of these relations lies in allowing complex mathematical structures to be analyzed and broken down into simpler, clearer components.

Operations on Sets

Operations on sets represent the dynamic dimension of set theory, as they allow for constructing entirely new sets from pre-existing ones:

- **Union** ($\cup$) :One of the most common operations; it expresses the gathering of elements from two or more sets into a single set that contains all elements belonging to any of them.
- **Intersection** ($\cap$) :Expresses the elements shared in common between sets. Therefore, it plays a vital role in revealing the points of convergence between different concepts or categories.

- **Difference ($\setminus$ or $-$)** :Allows for isolating elements that exist in one set but do not exist in another, making it an effective tool in processes of distinction and classification.
- **Complement (A' or $\setminus(A^{\{c\}})$)** :Involves identifying the elements that do not belong to a specific set within the framework of the universal set. This operation is deeply intertwined with the concept of negation in logic.

Together, these operations form a mathematical and logical foundation for understanding complex relationships between different groups.

Laws of Set Operations

Just as there are laws governing logical connectives in propositional logic, there are also laws regulating operations between sets:

- **Commutative Laws** :Show that the order of sets does not affect the outcome of a union or an intersection.
- **Associative Laws** :Allow for regrouping operations without changing the final results.
- **Distributive Laws** :Link union and intersection together.
- **Identity and Absorption Laws** :Play a major role in simplifying mathematical and logical expressions.
- **De Morgan's Laws** :These are among the most critical laws because they reveal the deep connection between set operations and logical connectives, demonstrating how complements can transform unions into intersections and vice versa.

Logical and Philosophical Importance of Set Theory

The significance of set theory is not restricted to mathematical mechanics; it extends directly into philosophy and logic. It represents a premier model for abstract thinking and the rational organization of knowledge, helping clarify concepts of classification, identity, difference, and belonging.

Analytic philosophy has leveraged the concepts of set theory to study language and the logical relations between concepts. Many contemporary

intellectual movements have also relied on it to analyze epistemological and structural frameworks. In mathematical logic, set theory has become an indispensable tool for building formal systems, formulating theorems, and analyzing relationships between propositions, predicates, and functions.

Applications of Set Theory

Today, set theory enters into a vast array of scientific and technical fields:

- **In Mathematics :**It is used to ground almost all modern mathematical branches.
- **In Computer Science :**It is utilized in designing relational databases, analyzing algorithms, and structuring programming languages.
- **In Artificial Intelligence :**It plays an important role in knowledge representation, expert systems, and logical networks.
- **In the Humanities and Social Sciences :**Concepts of sets are used in classification processes, statistical analysis, and studying relationships between different phenomena.

These applications demonstrate the universal character of the theory and its ability to cross the traditional boundaries between sciences.

Conclusion

Set theory represents one of the greatest intellectual achievements in the history of modern logic and mathematics, providing a precise language and an abstract methodology to organize knowledge and analyze relationships between things and concepts. Through the concepts of the set, the element, membership, containment, and various operations, it became possible to construct an expansive logical and mathematical framework that serves as the basis for much of contemporary science. Therefore, understanding set theory is not merely the study of a specific mathematical topic; rather, it is an essential entry point to grasping the deep structure of mathematical logic and its modern analytical tools.