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Subject: Mathematical Logic

Lecture Title: Introduction to Propositional Theory and Logical Operations

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Chapter Three: Propositional Theory

Introduction

Propositional theory is the foundation upon which modern mathematical logic is built, as propositions represent the primary units from which inferences, proofs, and various logical systems are constructed. Modern logicians have focused on studying propositions in terms of their structure, truth value, and the relationships that bind them, in order to build an accurate symbolic language that helps analyze human thought objectively.

The importance of propositional theory lies in the fact that it represents the starting point toward studying logical connectives, truth tables, and rules of inference, which are the topics that form the core of mathematical logic.

First: The Concept of a Proposition

A proposition in logic is a declarative sentence that can be judged as either true or false.

It is required that a proposition be a statement (informative), not a command, a question, or an exclamation.

Examples:

- The sun is shining.
- Water consists of hydrogen and oxygen.
- Baghdad is the capital of Iraq.

All of these statements are considered propositions because they can be judged as true or false.

However, the following expressions are not considered propositions:

- Open the door.
- Did you attend the lecture?.
- How beautiful spring is!.

This is because they do not carry a truth value.

Second: Elements of a Proposition

A proposition usually consists of three elements:

- **The Subject:** The thing we are talking about. For example, in "Man is rational", the subject is: Man.
- **The Predicate:** The characteristic or judgment that we affirm about the subject. In the previous example, the predicate is: rational.
- **The Copula:** The relationship that links the subject to the predicate, such as: is, to be, is not.

Third: The Truth Value of a Proposition

Every proposition has exactly one of two truth values:

- Truth (T)
- Falsity (F)

For example:

- The proposition: "Mosul is an Iraqi city" $\rightarrow$ Its truth value = True.
- The proposition: "Basra is the capital of Iraq" $\rightarrow$ Its truth value = False.

Truth is symbolized by the letter (T) and falsity by the letter (F).

Fourth: Types of Propositions

Propositions are divided into two main sections:

- **Simple Propositions:** These are propositions that do not contain any logical connective. Examples:
 - Man is mortal.
 - The Earth is spherical.
 - Water is liquid.
 - They are sometimes called atomic propositions.
- **Compound Propositions:** These are propositions that consist of two or more simple propositions linked together by logical connectives. Example:
 - The sun is shining and the weather is mild.
 - It consists of: "The sun is shining" and "The weather is mild", linked by the conjunction tool (and).

Fifth: Symbols of Propositions

Mathematical logic relies on symbols instead of natural language. Propositions are represented by letters: (P, Q, R, S) .

Example:

- (P) : Man is rational.
- (Q) : Man is mortal.

Through this, propositions can be expressed without the need to repeat their wording.

Sixth: The Principle of Bivalence

Traditional propositional logic is based on a fundamental principle called the Principle of Bivalence. It means that: every proposition is either true or false, and there is no third possibility. This principle is one of the most important foundations upon which truth tables are built.

Seventh: Characteristics of a Logical Proposition

A logical proposition must be:

1. **Declarative:** Meaning it carries an informative sense.

2. **Definite in Meaning:** So that it is not ambiguous.
3. **Judgeable:** Meaning it can be described as true or false.
4. **Logically Independent:** So that it has a clear, distinct content.

Eighth: Propositions in Natural Language and Symbolic Language

- **Natural Language:** "If the student works hard, then he will succeed."
- **Symbolic Language:** If we symbolize: (P) = The student works hard, and (Q) = The student succeeds, then the proposition becomes: $(P \rightarrow Q)$.

Symbolic language is characterized by: accuracy, brevity, clarity, and ease of analysis.

Ninth: Importance of Propositional Theory

The importance of propositional theory lies in the following:

- It represents the primary foundation of mathematical logic.
- It helps analyze arguments and inferences.
- It paves the way for studying logical connectives.
- It contributes to constructing truth tables.
- It is used in computer science and artificial intelligence.
- It helps detect cognitive and intellectual fallacies.

Tenth: Applications of Propositions in Philosophy

Philosophers use propositional theory in:

- Analyzing philosophical arguments.
- Studying the concepts of truth and falsity.
- Analyzing language.
- Examining rational proofs.
- Studying metaphysical and epistemological propositions.

It has become a central tool in contemporary analytic philosophy.

Eleventh: Practical Examples

- **The First Example:** The proposition: "Water freezes at zero degrees" is a simple proposition, and its truth value is True.
- **The Second Example:** The proposition: "The Earth is spherical and the sun revolves around the Earth" is a compound proposition consisting of two simple propositions.
- **The Third Example:** The proposition: "If the student attends the lecture, then he understands the lesson" represents a conditional proposition, which will be analyzed later in the logical connectives section.

Conclusion

The proposition is the fundamental unit in mathematical logic, representing every declarative statement that can be judged as true or false. It is divided into simple and compound propositions, and studying it is the first step toward understanding logical connectives, truth tables, and symbolic inferences.

Compound Propositions, Logical Connectives, Truth Tables, and Logical Inference

Introduction

After introducing the concept of logical propositions and their basic types, the study of mathematical logic moves to a more advanced stage, which is the analysis of compound propositions and the relations that arise between them. A logical proposition does not always remain in its simple form; it often links with other propositions through tools or logical connectives that lead to the formation of more complex structures. Hence, the need arose to study logical connectives, truth tables, rules of logical construction, and laws of inference, which represent the theoretical basis for proof in mathematical logic.

Compound propositions are the most common form in scientific and philosophical thinking because they allow for expressing various relationships between judgments and ideas. This is achieved by using logical connectives that link propositions together and give them new semantic meanings that differ from those of individual propositions.

The simplest of these connectives is Negation, which flips the truth value of a proposition; if a proposition is true, its negation becomes false, and if it is false, its negation becomes true. As for the Conjunction connective, it links two propositions using the tool (and), and for the resulting compound proposition to be true, both component propositions must be true. Conversely, the Disjunction connective allows for expressing different alternatives using the tool (or). The Conditional connective expresses a relation of implication between the antecedent and the consequent, while the Biconditional (Equivalence) connective represents the highest degree of logical connection because it indicates a mutual implication between two propositions.

The use of these connectives led to the emergence of what is known as compound propositions, which are structures consisting of more than one connected simple proposition within a specific logical framework. To verify the validity of these logical frameworks, logicians developed truth tables, which are an organized mathematical method that allows determining the truth value of any compound proposition, no matter how complex it may be.

Truth tables are based on the Principle of Bivalence, which assumes that every proposition is either true or false. By studying all possible combinations of values for the simple components, the final value of the compound proposition can be determined. This achievement represented a qualitative shift in the history of logic, as it allowed for the examination of arguments and inferences in a precise manner, free from subjective interpretations.

Through truth tables, different types of propositions can be distinguished. There are propositions that are always true regardless of the values of their components, known as Tautologies or logically true propositions. There are also propositions that are always false, which are called

Contradictions, in addition to Contingent propositions that may be true in some cases and false in others.

The analysis of compound propositions led to the discovery of a set of fundamental logical laws governing the construction of inferences, the most important of which are the Law of Identity, the Law of Non-Contradiction, and the Law of Excluded Middle. Other laws emerged, such as De Morgan's Laws, which explain how to deal with negation in compound propositions, alongside the laws of Distribution, Commutation, and Absorption, which allow for simplifying logical formulas and transforming them into equivalent forms.

The study of these laws is not limited to the theoretical aspect; rather, it represents an essential basis in logical proof. A proof in mathematical logic is the process of deriving a conclusion from certain premises according to specific rules of inference. The importance of proof lies in the fact that it provides an objective means to uncover the validity or invalidity of results.

Among the most famous rules of inference is the *Modus Ponens* (Rule of Detachment), which allows moving from a conditional proposition and a true antecedent premise to the resulting conclusion. There is also the rule of Hypothetical Syllogism, Conjunction, Simplification, and other rules that form the structural backbone of logical inference.

Logical inference means moving from known propositions to a new proposition that necessarily follows from them. The validity of an inference does not depend on the content of the propositions but rather on the logical form that links them. Therefore, mathematical logic is more concerned with form than content, as it seeks to uncover the formal structure of human thought.

The importance of these topics stands out particularly in contemporary philosophy, where the analysis of philosophical arguments has come to rely heavily on the tools of symbolic logic. Furthermore, computer science and artificial intelligence depend on logical connectives and inference rules to design algorithms, build expert systems, and create smart databases.

Thus, it is clear that the study of compound propositions, logical connectives, truth tables, and laws of inference represents the heart of modern mathematical logic. It provides the necessary tools to analyze human thinking, evaluate arguments, and uncover the logical relationships underlying various forms of scientific and philosophical knowledge. Without these tools, it becomes difficult to build rigorous proofs or verify the validity of inferences upon which organized thinking in various fields of knowledge relies.