Example: Evaluate the double integral $\int_0^2 \int_0^{2x} xy\,dydx$

Inner integration

$$\int_0^{2x} xy\,dy = x\left[\frac{y^2}{2}\right]_0^{2x} = \frac{x}{2}[y^2]_0^{2x} = \frac{x}{2}((2x)^2 - 0) = \frac{x}{2}((2x)^2 - 0) = 2x^3$$

Outer integration

$$2\int_0^2 x^3\,dx = 2\left[\frac{x^4}{4}\right]_0^2 = 2\left(\frac{2^4}{4} - \frac{0^4}{4}\right) = 2(4) = 8$$

$$\therefore \int_0^2 \int_0^{2x} xy\,dydx = 8$$

Example: Evaluate the double integral $\int_0^1 \int_{y^2}^{y} \frac{e^y}{y}\,dxdy$

Inner integration

$$\int_{y^2}^{y} \frac{e^y}{y}\,dx = \frac{e^y}{y}[x]_{y^2}^{y} = \frac{e^y}{y}(y - y^2) = e^y - ye^y$$

Outer integration

$$\int_0^1 (e^y - ye^y)dy = \int_0^1 e^y dy - \int_0^1 ye^y dy = [e^y]_0^1 - \int_0^1 ye^y dy = e - 1 - \int_0^1 ye^y dy$$

التكامل الأخير يمكن تكامله بطريقة التجزئة: نفرض ان $dv = e^y dy$ ، $u = y$. أذاً

$du = dy$, $v = e^y$

$$\int_0^1 ye^y dy = [ye^y]_0^1 - \int_0^1 e^y dy = [ye^y]_0^1 - [e^y]_0^1 = [e^1 - 0] - [e^1 - e^0] = 1$$

$$\therefore \int_0^1 \int_{y^2}^{y} \frac{e^y}{y} dxdy = e - 1 - \int_0^1 ye^y dy = e - 1 - 1 = e - 2$$

مثال: أحسب قيمة التكامل $\iint_R (3y^2 - x)\,dxdy$ حيث أن $R = \{(x, y): 0 \le x \le 2\,, 1 \le y \le 2\}$

الحل: التكامل الداخلي

$$\int_0^2 (3y^2 - x)\,dx = \left[3y^2 x - \frac{x^2}{2}\right]_0^2 = (6y^2 - 2) - (0) = 6y^2 - 2$$

التكامل الخارجي: $\int_1^2 (6y^2 - 2)\,dy = [2y^3 - 2y]_1^2 = (16 - 4) - (2 - 2) = 12$

$$\therefore \int_1^2 \int_0^2 (3y^2 - x)\,dxdy = 12$$

Example: Evaluate the double integral $\int_1^2 \int_y^{y^2} dx dy$

Inner integration

$$\int_y^{y^2} dx = [x]_y^{y^2} = (y^2 - y)$$

Outer integration

$$\int_1^2 (y^2 - y)\, dy = \left[\frac{y^3}{3} - \frac{y^2}{2}\right]_1^2 = \left(\frac{2^3}{3} - \frac{2^2}{2}\right) - \left(\frac{1^3}{3} - \frac{1^2}{2}\right) = \left(\frac{8}{3} - \frac{4}{2}\right) - \left(\frac{1}{3} - \frac{1}{2}\right) = \frac{5}{6}$$

$$\therefore \int_1^2 \int_y^{y^2} dx dy = \frac{5}{6}$$

Example: Evaluate the double integral $\int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx dy$

Inner integration

$$\int_0^{\ln y} e^{x+y}\, dx = \int_0^{\ln y} e^y\, e^x\, dx = e^y [e^x]_0^{\ln y} = e^y \left[e^{\ln y} - e^0\right] = e^y [y - 1] = y e^y - e^y$$

Outer integration

$$\int_1^{\ln 8} (y e^y - e^y) dy = \int_1^{\ln 8} y\, e^y dy - \int_1^{\ln 8} e^y dy$$

الجزء الملون يكامل بطريقة التجزئة:

$$\int_1^{\ln 8} y e^y dy = [y e^y]_1^{\ln 8} - \int_1^{\ln 8} e^y dy = [y e^y]_1^{\ln 8} - [e^y]_1^{\ln 8}$$

$$= \left(\ln 8\, e^{\ln 8} - e^1\right) - \left(e^{\ln 8} - e^1\right) = 8 \ln 8 - e - 8 + e = 8 \ln 8 - 8$$

أما تكامل الجزء الثاني يكون $\int_1^{\ln 8} e^y dy = [e^y]_1^{\ln 8} = e^{\ln 8} - e^1 = 8 - e$

$$\therefore \int_1^{\ln 8} \int_0^{\ln y} e^{x+y}\, dx dy = \int_1^{\ln 8} y\, e^y dy - \int_1^{\ln 8} e^y dy = 8 \ln 8 - 8 - (8 - e)$$

$$= 8 \ln 8 - 16 + e$$

Homework

Evaluate each of the following double integrals:

$$\int_{-1}^{2}\int_{x^2}^{x+2} \mathrm{d}y\mathrm{d}x \quad , \quad \int_{-1}^{3}\int_{-4}^{1} \mathrm{d}y\mathrm{d}x \quad , \quad \int_{0}^{2}\int_{0}^{\sqrt{4-y^2}} \mathrm{d}x\mathrm{d}y \quad , \quad \int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} \sqrt{9-x^2-y^2}\,\mathrm{d}x\mathrm{d}y$$

$$\int_{0}^{3}\int_{0}^{2} 12x^2y^3\,\mathrm{d}y\mathrm{d}x \quad , \quad \int_{0}^{1}\int_{\sqrt{x}}^{1} \sqrt{1+y^3}\,\mathrm{d}y\mathrm{d}x \quad , \quad \int_{0}^{4}\int_{\frac{\sqrt{x}}{2}}^{1} e^{y^3}\,\mathrm{d}y\mathrm{d}x \quad , \quad \int_{0}^{4}\int_{\sqrt{y}}^{\sqrt{8-y}} \frac{1}{(1+y)^2}\,\mathrm{d}x\mathrm{d}y$$

$$\int_{0}^{1}\int_{x^2}^{1} x^3\sin(y^3)\,\mathrm{d}y\mathrm{d}x \quad , \quad \int_{0}^{1}\int_{0}^{y} 2xe^{-y}\mathrm{d}x\mathrm{d}y \quad , \quad \int_{0}^{1}\int_{0}^{x} \frac{\sin x}{x}\mathrm{d}y\mathrm{d}x \quad , \quad \int_{0}^{\pi}\int_{0}^{\sin x} y\,\mathrm{d}y\mathrm{d}x$$

$$\int_{0}^{3}\int_{\frac{2}{3}y}^{2} x^2y\,\mathrm{d}x\mathrm{d}y \quad , \quad \int_{0}^{1}\int_{x}^{\sqrt{x}} (4x+10y)\,\mathrm{d}y\mathrm{d}x$$

Applications of Double Integrals

تطبيقات التكامل الثنائي

يمكن استخدام التكامل الثنائي في العديد من المجالات مثل الرياضيات والفيزياء والهندسة. حيث يمكن استخدامه لحساب المساحات والحجوم والكتلة والعزم والمراكز الثقلية والعديد من الخصائص الأخرى للأشكال الهندسية ثنائية الأبعاد.

حساب الحجم

أذا كانت $\mathrm{z} = f(x, \mathrm{y})$ تمثل معادلة السطح فأن:

$$\mathrm{V} = \iint_{\mathrm{R}} f(x, \mathrm{y})\, \mathrm{dA}$$

تعطي حجم المجسم الواقع بين السطح والمستوى $x\mathrm{y}$.

مثال: أوجد الحجم تحت السطح $\mathrm{z} = x + 2\mathrm{y}$ وفوق المستطيل R حيث أن:

$$\mathrm{R} = \{(x, \mathrm{y}): 1 \le x \le 2, 3 \le \mathrm{y} \le 5\}$$

الحل:

$$\mathrm{V} = \iint_{\mathrm{R}} (x + 2\mathrm{y})\, \mathrm{dy}\mathrm{d}x = \int_{1}^{2} \int_{3}^{5} (x + 2\mathrm{y})\, \mathrm{dy}\mathrm{d}x$$

التكامل الداخلي:

$$\int_{3}^{5} (x + 2\mathrm{y})\, \mathrm{dy} = \left[x\mathrm{y} + 2\frac{\mathrm{y}^2}{2}\right]_{3}^{5} = [x\mathrm{y} + \mathrm{y}^2]_{3}^{5} = (5x + 5^2) - (3x + 3^2)$$

$$= 5x + 25 - 3x - 9 = 2x + 16$$

التكامل الخارجي:

$$\int_{1}^{2} (2x + 16)\, \mathrm{d}x = \left[2\frac{x^2}{2} + 16x\right]_{1}^{2} = [x^2 + 16x]_{1}^{2}$$

$$= \big(2^2 + 16(2)\big) - \big(1^2 + 16(1)\big) = 36 - 17 = 19$$

$$\therefore \mathrm{V} = \int_{1}^{2} \int_{3}^{5} (x + 2\mathrm{y})\, \mathrm{dy}\mathrm{d}x = 19$$

مثال: جد حجم الجسم المحدد بالسطوح الاتية:

$x = 2 \ , \ \mathrm{z} = 0 \ , \ \mathrm{y} = 0 \ , \ x^2 = \mathrm{y} + \mathrm{z}$

الحل:

$$\mathrm{V} = \iint_{\mathrm{R}} (x^2 - \mathrm{y}) \, \mathrm{d}y\mathrm{d}x = \int_0^2 \int_0^{x^2} (x^2 - \mathrm{y}) \, \mathrm{d}y\mathrm{d}x = \int_0^2 \left[x^2\mathrm{y} - \frac{1}{2}\mathrm{y}^2 \right]_0^{x^2} \mathrm{d}x$$

$$= \int_0^2 \left(\left(x^2(x^2) - \frac{1}{2}(x^2)^2 \right) - (0) \right) \mathrm{d}x = \int_0^2 \frac{x^4}{2} \mathrm{d}x = \left[\frac{x^5}{10} \right]_0^2 = \left[\frac{x^5}{10} \right]_0^2 = \frac{2^5}{10} = \frac{16}{5}$$

$$\therefore \mathrm{V} = \int_0^2 \int_0^{x^2} (x^2 - \mathrm{y}) \, \mathrm{d}y\mathrm{d}x = \frac{16}{5}$$

مثال: أوجد حجم المجسم تحت السطح $\mathrm{z} = x^2 + \mathrm{y}^2$ وفوق المنطقة R حيث أن:

$$\mathrm{R} = \{(x, \mathrm{y}) : \mathrm{y}^2 \le x \le \sqrt{\mathrm{y}}, 0 \le \mathrm{y} \le 1\}$$

الحل:

$$\mathrm{V} = \iint_{\mathrm{R}} (x^2 + \mathrm{y}^2) \, \mathrm{d}x\mathrm{d}\mathrm{y} = \int_0^1 \int_{\mathrm{y}^2}^{\sqrt{\mathrm{y}}} (x^2 + \mathrm{y}^2) \, \mathrm{d}x\mathrm{d}\mathrm{y} = \int_0^1 \left[\frac{x^3}{3} + \mathrm{y}^2 x \right]_{\mathrm{y}^2}^{\sqrt{\mathrm{y}}} \mathrm{d}\mathrm{y}$$

$$= \int_0^1 \left(\left(\frac{\left(\sqrt{\mathrm{y}}\right)^3}{3} + \mathrm{y}^2\sqrt{\mathrm{y}} \right) - \left(\frac{(\mathrm{y}^2)^3}{3} + \mathrm{y}^2\mathrm{y}^2 \right) \right) \mathrm{d}\mathrm{y}$$

$$= \int_0^1 \left(\left(\frac{\mathrm{y}^{\frac{3}{2}}}{3} + \mathrm{y}^{\frac{5}{2}} \right) - \left(\frac{\mathrm{y}^6}{3} + \mathrm{y}^4 \right) \right) \mathrm{d}\mathrm{y} = \left[\left(\frac{\mathrm{y}^{\frac{3}{2}+1}}{3\left(\frac{3}{2}+1\right)} + \frac{\mathrm{y}^{\frac{5}{2}+1}}{\frac{5}{2}+1} \right) - \left(\frac{\mathrm{y}^7}{21} + \frac{\mathrm{y}^5}{5} \right) \right]_0^1$$

$$\left(\frac{(1)^{\frac{3}{2}+1}}{3\left(\frac{3}{2}+1\right)} + \frac{(1)^{\frac{5}{2}+1}}{\frac{5}{2}+1} \right) - \left(\frac{(1)^7}{21} + \frac{(1)^5}{5} \right) - (0) = \frac{2}{15} + \frac{2}{7} - \frac{1}{21} - \frac{1}{5}$$

$$= \frac{2-3}{15} + \frac{6-1}{21} = \frac{-1}{15} + \frac{5}{21} = \frac{-7+25}{105} = \frac{18}{105}$$

حساب المساحة

أذا كانت $f(x,\mathrm{y})=1$ فأن:

$$\mathrm{A(R)}=\iint_{\mathrm{R}}\mathrm{dA}$$

حيث ان $\mathrm{A(R)}$ تمثل مساحة المنطقة المغلقة R .

مثال: لتكن $f:\mathbb{R}^2\to\mathbb{R}$ معرفة كالاتي

$$f(x,\mathrm{y})=\sqrt{1-x-\mathrm{y}}$$

عين مجال الدالة f واحسب مساحة المنطقة R حيث ان:

$$\mathrm{R}=\{(x,\mathrm{y})\in\mathbb{R}^2: x\geq 0,\mathrm{y}\geq 0\}$$

الحل:

$$\mathrm{D}_f=\{(x,\mathrm{y})\in\mathbb{R}^2: 1-x-\mathrm{y}\geq 0\}=\{(x,\mathrm{y})\in\mathbb{R}^2: \mathrm{y}\leq 1-x\}$$

وهي مجموعة نقاط المستوي الواقعة تحت المستقيم $\mathrm{y}=1-x$.

لحساب مساحة المنطقة R :

$$\mathrm{A(R)}=\iint_{\mathrm{R}}\mathrm{dy}dx=\int_0^1\left(\int_0^{1-x}\mathrm{dy}\right)\mathrm{d}x=\int_0^1(1-x)\,\mathrm{d}x=\left[x-\frac{x^2}{2}\right]_0^1=\frac{1}{2}$$

مثال: لتكن $f:\mathbb{R}^2\to\mathbb{R}$ معرفة كالاتي

$$f(x,\mathrm{y})=\sqrt{1+x-\mathrm{y}}$$

عين مجال الدالة f واحسب مساحة المنطقة R حيث ان:

$$\mathrm{R}=\{(x,\mathrm{y})\in\mathbb{R}^2: x\leq 0,\mathrm{y}\geq 0\}$$

الحل:

$$\mathrm{D}_f=\{(x,\mathrm{y})\in\mathbb{R}^2: 1+x-\mathrm{y}\geq 0\}=\{(x,\mathrm{y})\in\mathbb{R}^2: \mathrm{y}\leq 1+x\}$$

لحساب مساحة المنطقة R

$$\mathrm{A(R)}=\iint_{\mathrm{R}}\mathrm{dy}dx=\int_{-1}^0\left(\int_0^{1+x}\mathrm{dy}\right)\mathrm{d}x=\int_{-1}^0(1+x)\,\mathrm{d}x=\left[x+\frac{x^2}{2}\right]_{-1}^0$$

$$=(0)-\left(-1+\frac{(-1)^2}{2}\right)=-\left(\frac{-1}{2}\right)=\frac{1}{2}$$

مثال: لتكن $f:\mathbb{R}^2 \to \mathbb{R}$ معرفة كالاتي

$$f(x,\mathrm{y}) = \sqrt{\mathrm{y} - x}$$

عين مجال الدالة f واحسب مساحة المنطقة R حيث ان:

$$\mathrm{R} = \{(x,\mathrm{y}) \in \mathbb{R}^2 : x \geq 0, \mathrm{y} \leq 1\}$$

الحل:

$$\mathrm{D}_f = \{(x,\mathrm{y}) \in \mathbb{R}^2 : \mathrm{y} - x \geq 0\} = \{(x,\mathrm{y}) \in \mathbb{R}^2 : \mathrm{y} \geq x\}$$

هي مجموعة نقاط المستوي الواقعة فوق المستقيم فوق المستقيم $\mathrm{y} = x$

لحساب مساحة المنطقة R

$$\mathrm{A(R)} = \iint_{\mathrm{R}} \mathrm{d}y\mathrm{d}x = \int_0^1 \left(\int_x^1 \mathrm{dy} \right) \mathrm{d}x = \int_0^1 (1-x)\,\mathrm{d}x = \left[x - \frac{x^2}{2} \right]_0^1 = \frac{1}{2}$$

Example: Find the area bounded by the curve $\mathrm{y} = x^2$ and $x-$ axis and the line $x = 0\ ,\ x = 2.$

Solution:

$$\mathrm{A(R)} = \int_0^2 \int_0^{x^2} \mathrm{d}y\mathrm{d}x = \int_0^2 [\mathrm{y}]_0^{x^2}\,\mathrm{d}x = \int_0^2 x^2\,\mathrm{d}x = \left[\frac{x^3}{3} \right]_0^2 = \frac{8}{3}$$

Example: Find the area bounded between $\mathrm{y} = x^2$ and $\mathrm{y} = 2x.$

Solution:

$$x^2 = 2x \Rightarrow x^2 - 2x = 0 \Rightarrow x(x-2) = 0 \quad \Rightarrow \quad x = 0\ , \qquad x = 2$$

$$\mathrm{A(R)} = \int_0^2 \int_{x^2}^{2x} \mathrm{d}y\mathrm{d}x = \int_0^2 [\mathrm{y}]_{x^2}^{2x}\,\mathrm{d}x = \int_0^2 (2x - x^2)\,\mathrm{d}x = \left[x^2 - \frac{x^3}{3} \right]_0^2 = 4 - \frac{8}{3} = \frac{4}{3}$$

Example: Find the area bounded between $\mathrm{y} = 8 - \frac{x^2}{2}$ and $\mathrm{y} = 2 - \frac{x}{2}$

Solution:

$$8 - \frac{x^2}{2} = 2 - \frac{x}{2} \Rightarrow 8 - \frac{x^2}{2} - 2 + \frac{x}{2} = 0 \Rightarrow 12 + x - x^2 = 0 \Rightarrow x^2 - x - 12 = 0$$

$$\Rightarrow\ (x-4)(x+3) = 0 \quad \Rightarrow \quad x = 4\ ,\ x = -3$$

$$\mathrm{A(R)} = \int_{-3}^{4}\int_{2-\frac{x}{2}}^{8-\frac{x^2}{2}} \mathrm{d}y\mathrm{d}x = \int_{-3}^{4}[\mathrm{y}]_{2-\frac{x}{2}}^{8-\frac{x^2}{2}}\,\mathrm{d}x = \int_{-3}^{4}\left(\left(8-\frac{x^2}{2}\right)-\left(2-\frac{x}{2}\right)\right)\mathrm{d}x$$

$$= \int_{-3}^{4}\left(8-\frac{x^2}{2}-2+\frac{x}{2}\right)\mathrm{d}x = \int_{-3}^{4}\left(6-\frac{x^2}{2}+\frac{x}{2}\right)\mathrm{d}x = \left[6x-\frac{x^3}{6}+\frac{x^2}{4}\right]_{-3}^{4}$$

$$= \left(6(4)-\frac{4^3}{6}+\frac{4^2}{4}\right)-\left(6(-3)-\frac{(-3)^3}{6}+\frac{(-3)^2}{4}\right)$$

$$= 24-\frac{64}{6}+4+18-\frac{27}{6}+\frac{9}{4} = 46-\frac{37}{6}+\frac{9}{4} = \frac{552-74+27}{12} = \frac{505}{12}$$

Example: Find the area bounded between $\mathrm{y} = x^3$ and $\mathrm{y} = x$.

الحل: نوجد اولاً نقاط تقاطع المنحني مع المستقيم

$$x^3 = x \Longrightarrow x^3 - x = 0 \Longrightarrow x(x^2-1) = 0 \Longrightarrow x(x-1)(x+1) = 0$$

$$\Longrightarrow x = 0, -1, 1$$

ومن هذه القيم لـــ x نحصل على قيم y :

$$\mathrm{y} = 0, -1, 1$$

أذا المعادلتان تتقاطعان عند النقاط $(-1,-1), (0,0), (1,1)$.

$$\therefore\ \mathrm{A(R)} = \int_{-1}^{0}\int_{x}^{x^3} \mathrm{d}y\mathrm{d}x + \int_{0}^{1}\int_{x^3}^{x} \mathrm{d}y\mathrm{d}x$$

$$= \int_{-1}^{0}[\mathrm{y}]_{x}^{x^3}\,\mathrm{d}x + \int_{0}^{1}[\mathrm{y}]_{x^3}^{x}\,\mathrm{d}x = \int_{-1}^{0}(x^3-x)\,\mathrm{d}x + \int_{0}^{1}(x-x^3)\,\mathrm{d}x$$

$$= \left[\frac{x^4}{4}-\frac{x^2}{2}\right]_{-1}^{0} + \left[\frac{x^2}{2}-\frac{x^4}{4}\right]_{0}^{1} = \left(\frac{0^4}{4}-\frac{0^2}{2}\right)-\left(\frac{(-1)^4}{4}-\frac{(-1)^2}{2}\right)$$

$$+\left(\frac{1^2}{2}-\frac{1^4}{4}\right)-\left(\frac{0^2}{2}-\frac{0^4}{4}\right)$$

$$= -\frac{1}{4}+\frac{1}{2}+\frac{1}{2}-\frac{1}{4} = \frac{1}{2}$$