Sol.: We find the inverse matrix

$$(X'X)^{-1} = \begin{bmatrix} 9.1667 & -0.5 & -0.833 & -2.333 \\ & 0.25 & 0 & 0 \\ & & 0.1667 & 0.1667 \\ & & & 0.75 \end{bmatrix}$$

Thus, the vector of estimated parameters is:

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{bmatrix} = (X'X)^{-1}X'y = \begin{bmatrix} -6 \\ 1 \\ 3 \\ 1.833 \end{bmatrix}$$

We now estimate the value of the linear function $L = \beta_2 - \beta_3$ as follows:

$$L = \beta_2 - \beta_3$$

whereas:

$$\hat{\beta}_2 = 3 \,, \hat{\beta}_3 = 1.833$$

The variance of the linear function is:

$$S^2_{(\hat{\beta}_2 - \hat{\beta}_3)} = S^2_{\hat{\beta}_2} + S^2_{\hat{\beta}_3} - 2Cov(\hat{\beta}_2, \hat{\beta}_3)$$

$$S^2_{(\hat{\beta}_2 - \hat{\beta}_3)} = MSe(C_{22} + C_{33} - 2C_{23})$$

$$S^2_{(\hat{\beta}_2 - \hat{\beta}_3)} = 1.22\big(0.1667 + 0.75 - (2)(0.1667)\big)$$

$$S^2_{(\hat{\beta}_2 - \hat{\beta}_3)} = 0.7116$$

$$S_{\hat{L}} = S_{\hat{\beta}_2 - \hat{\beta}_3} = \sqrt{0.7166} = 0.8436$$

To find the confidence interval we use the following formula:

$$\hat{L} - t_{\frac{1}{2}\alpha,n-m-1} S_{\hat{L}} \leq L \leq \hat{L} + t_{\frac{1}{2}\alpha,n-m-1} S_{\hat{L}}$$

$$\left(\hat{\beta}_2 - \hat{\beta}_3\right) - t_{\frac{1}{2}0.05,7-3-1} S_{\hat{L}} \leq L \leq \left(\hat{\beta}_2 - \hat{\beta}_3\right) + t_{\frac{1}{2}0.05,7-3-1} S_{\hat{L}}$$

$$(3 - 1.833) - (3.182)(0.8436) \leq L \leq (3 - 1.833) + (3.182)(0.8436)$$

$$-1.52 \leq L \leq 3.85$$

$$-1.52 \leq \beta_2 - \beta_3 \leq 3.85$$

Since the period contains zero, there is no significant difference between β_2 and β_3.

Estimate a confidence interval for the mean response

Assume that $X_{01}, X_{02}, \dots, X_{0m}$ It is one of the values of the independent variables $X_1, X_2, \dots, X_m$ We want to estimate a confidence interval for the average response at a certain point, which is $\bar{y}_x$ which equals:

$$\bar{y}_x = \hat{y}_0 = X_0' \hat{\beta}$$

whereas:

$$X_0 = \begin{bmatrix} 1 \\ X_{01} \\ X_{02} \\ \vdots \\ X_{0m} \end{bmatrix}$$

That is valuable $\bar{y}_x$ It follows the normal distribution with an arithmetic mean of:

$$E\left(\frac{y}{X_0}\right) = \hat{\beta}_0 X_0 + \hat{\beta}_1 X_1 + \dots + \hat{\beta}_m X_m$$

And the variance

$$\sigma^2_{\hat{y}_X} = \frac{\hat{\sigma}^2}{w} = \frac{MSe}{w}$$

Since:

$$\frac{1}{w} = X_0'(X'X)^{-1} X_0$$

Therefore, the confidence interval for the mean response is:

$$\hat{y}_0 - t_{\frac{1}{2}\alpha, n-m-1} \sqrt{\sigma^2_{\hat{y}_X}} \leq y_0 \leq \hat{y}_0 + t_{\frac{1}{2}\alpha, n-m-1} \sqrt{\sigma^2_{\hat{y}_X}}$$

Example:

From the following data, find a 95% confidence interval for the mean response at $X_1=3$, $X_2=4$, $X_3=5$ and $t_{\frac{1}{2}0.05,7-3-1} = t_{0.025,3} = 3.182$ The information is as follows:

$$\hat{\beta} = \begin{bmatrix} -6 \\ 1 \\ 3 \\ 1.833 \end{bmatrix}, MSe = 1.22$$

$$(X'X)^{-1} = \begin{bmatrix} 9.1667 & -0.5 & -0.833 & -2.333 \\ & 0.25 & 0 & 0 \\ & & 0.1667 & 0.1667 \\ & & & 0.75 \end{bmatrix}$$

solution:

we have:

$$\hat{y}_0 = X_0'\hat{\beta}$$

We compensate for values X ($X_1=3$, $X_2=4$, $X_3=5$) in X_0' , and about values $\hat{\beta}$ Their estimated values are as follows:

$$\hat{y}_0 = X_0'\hat{\beta} = \begin{bmatrix} 1 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -6 \\ 1 \\ 3 \\ 1.833 \end{bmatrix} = 18.1665$$

The variance $\hat{y}_0$ is $\frac{MSe}{w}$ whereas:

$$\frac{1}{w} = X_0'(X'X)^{-1}X_0$$

$$X_0'(X'X)^{-1}X_0 = \begin{bmatrix} 1 & 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 9.1667 & -0.5 & -0.833 & -2.333 \\ -0.5 & 0.25 & 0 & 0 \\ -0.833 & 0 & 0.1667 & 0.1667 \\ -2.333 & 0 & 0.1667 & 0.75 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$