4.5 : Some Discrete Distribution to Find

« Expectation, Variance, Moment Generating Function »

A - $E(X) = \sum_{\forall x} x\, p(x)$

B - $Var(X) = E(X - E(X))^2 = E(X^2) - (E(X))^2$

C - $M_X(t) = m_X(t) = E(e^{tX}) = \sum e^{tx} p(x)$

1- Uniform Distribution :-

$p(x) = \frac{1}{n}$; $x = 1, 2, 3, \cdots, n$; $X \sim Ud(n)$

A- $E(X) = \sum_{\forall x} x\, p(x) = \sum_{x=1}^{n} x \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x = \frac{1}{n} \cdot \frac{n(n+1)}{2}$

$\therefore E(X) = \frac{n+1}{2}$

B- $Var(X) = \sigma_X^2 = E(X^2) - [E(X)]^2$

$\therefore E(X^2) = \sum_{x=1}^{n} x^2 \frac{1}{n} = \frac{1}{n} \sum_{x=1}^{n} x^2 = \frac{1}{n} * \frac{n(n+1)(2n+1)}{6}$

$\therefore E(X^2) = \frac{(n+1)(2n+1)}{6}$

$\therefore Var(X) = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{2(2n^2+n+2n+1) - 3n^2 - 6n - 3}{12}$

$= \frac{4n^2 + 6n + 2 - 3n^2 - 6n - 3}{12} = \frac{n^2 - 1}{12}$

C- $M_X(t) = m_X(t) = E(e^{tX}) = \sum_{x=1}^{n} e^{tx} \frac{1}{n}$

$= \frac{1}{n} \sum_{x=1}^{n} e^{tx} = \frac{1}{n} [e^{t} + e^{2t} + e^{3t} + \cdots + e^{nt}]$

we have $\sum_{x=0}^{n} \lambda^x = \frac{1-\lambda^{n+1}}{1-\lambda}$ متوالية هندسية منتهية

$\therefore m_X(t) = \frac{e^t}{n} [1 + e^{t} + e^{2t} + \cdots + e^{(n-1)t}] = \frac{e^t}{n} \sum_{x=0}^{n-1} e^{tx}$

$= \frac{e^t}{n} \left[\frac{1-(e^t)^n}{1-e^t}\right] = \frac{e^t}{n} \left[\frac{1-e^{nt}}{1-e^t}\right]$

$$m_X^{k}(0) = \left.\frac{d^{k} m_X(t)}{dt^{k}}\right|_{t=0} = m_k$$

$$\therefore m_X'(0) = m_1 = \frac{n+1}{2} \; ; \; m_X''(0) = m_2 = \frac{(n+1)(2n+1)}{6}$$

$$\therefore Var(x) = m_2 - m_1^2 = \frac{n^2-1}{12}$$

2- Bernoulli Distribution

$$P(x) = p^{x} q^{1-x} \quad ; \quad X = 0,1 \quad ; \quad X \sim Ber(p)$$

A- $E(x) = \sum_{x=0}^{1} x\,p(x) = \sum_{x=0}^{1} x\,p^{x} q^{1-x} = 0 * p^{0} q^{1-0} + 1 * p^{1} q^{1-1} = \underline{\underline{p}}$

$\therefore E(x) = p$

B- $Var(x) = E(x^2) - [E(x)]^2$

$\therefore E(x^2) = \sum x^2 p(x) = \sum_{x=0}^{1} x^2 p^{x} q^{1-x} = 0^2 * p^{0} q^{1} + 1^2 * p^{1} q^{0} = \underline{\underline{p}}$

$\therefore Var(x) = p - p^2 = p(1-p) = \underline{\underline{pq}}$

C- $M_x(t) = m_x(t) = E(e^{tx}) = \sum_{x=0}^{1} e^{tx} p^{x} q^{1-x} = e^{0} p^{0} q^{1} + e^{t} p^{1} q^{0}$

$\therefore m_x(t) = q + pe^{t}$

3- Binomial Distribution :-

$$P(x) = C_x^n p^{x} q^{n-x} \quad ; \quad X = 0,1,2,\dots,n \quad ; \quad X \sim B(n,p)$$

A- $E(x) = \sum_{\forall x} x\,p(x) = \sum_{x=0}^{n} x\, C_x^n p^{x} q^{n-x} = \sum_{x=0}^{n} x \frac{n!}{x!(n-x)!} p^{x} q^{n-x}$

$= \sum_{x=0}^{n} \cancel{x} \frac{n(n-1)!}{\cancel{x}(x-1)!(n-x)!} p\, p^{x-1} q^{n-x}$

$= np \sum_{x=0}^{n-1} \frac{(n-1)!}{(x-1)!(n-x)!} p^{x-1} q^{n-x} = np(p+q)^{n-1}$

$\therefore E(x) = np(1)^{n-1} = \underline{np}$

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B- $Var(x) = E(x^2) - [E(x)]^2$

$$= E\,x(x-1) + E(x) - [E(x)]^2$$

$$\therefore E\,x(x-1) = \sum_{\forall x} x(x-1)\, C^n_x\, p^x q^{n-x} = \sum x(x-1) \frac{n!}{x!(n-x)!} p^x q^{n-x}$$

$$= \sum_{\forall x} \frac{x(x-1)\, n(n-1)(n-2)!}{x(x-1)(x-2)!\,(n-x)!}\, p^2 p^{x-2} q^{n-x}$$

$$= n(n-1)p^2 \sum_{\forall x} \frac{(n-2)!}{(x-2)!\,(n-x)!}\, p^{x-2} q^{n-x}$$

$$= n(n-1)p^2 (p+q)^{n-2} = n(n-1)p^2 (1)^{n-2}$$

$$\therefore Var(x) = n^2p^2 - np^2 + np - n^2p^2 = np - np^2$$

$$= np(1-p) = npq$$

C- $\mathcal{M}_x(t) = m_x(t) = E(e^{tx}) = \sum_{\forall x} e^{tx}\, C^n_x\, p^x q^{n-x}$

$$= \sum C^n_x (pe^t)^x q^{n-x} \quad ; \quad pe^t = a, \quad q = b$$

$$(a+b)^n = \sum C^n_x a^x b^{n-x}$$

$$\therefore \mathcal{M}_x(t) = m_x(t) = (pe^t + q)^n$$

4- Poisson Distribution :

$$P(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad , \quad x = 0, 1, 2, \ldots, \quad x \sim P_o(\lambda)$$

A- $E(x) = \sum_{\forall x} x\, p(x) = \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} x \frac{\lambda \lambda^{x-1}}{x(x-1)!}$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!} \quad ; \quad e^{\lambda} = \sum_{x=0}^{\infty} \frac{\lambda^x}{x!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda e^0 = \lambda$$

$$\therefore E(x) = \lambda$$

B- $Var(x) = \sigma_x^2 = E x^2 - [Ex]^2 = E x(x-1) + Ex - [Ex]^2$

$\therefore E x(x-1) = \sum_{x=0}^{\infty} x(x-1) \frac{e^{-\lambda}\lambda^x}{x!} = \sum \frac{x(x-1) e^{-\lambda} \lambda^2 \lambda^{x-2}}{x(x-1)(x-2)!}$

$= \lambda^2 e^{-\lambda} \sum_x \frac{\lambda^{x-2}}{(x-2)!} = \lambda^2 e^{-\lambda} e^{\lambda} = \lambda^2$

$\therefore Var(x) = \lambda^2 + \lambda - \lambda^2 = \underline{\underline{\lambda}}$

C- $M_x^t = m_x(t) = E(e^{tx}) = \sum_{x=0} e^{tx} \frac{e^{-\lambda}\lambda^x}{x!}$

$= \sum_{x=0}^{\infty} \frac{e^{-\lambda}(\lambda e^t)^x}{x!} = e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!} = e^{-\lambda} e^{\lambda e^t}$

$\therefore m_x(t) = e^{\lambda(e^t - 1)}$

5- Geometric Distribution :-

$P(x) = pq^x \quad , \quad x = 0, 1, 2, 3, \dots \quad ; \quad x \sim Ge(p)$

A- $E(x) = \sum_{x=0}^{\infty} x p q^x = p \sum_{x=0}^{\infty} x q^x = p[0 q^0 + 1q^1 + 2q^2 + \dots]$

$= p[q + 2q^2 + 3q^3 + 4q^4 + \dots]$

$= pq[1 + 2q^1 + 3q^2 + 4q^3 + \dots]$

$= pq \sum_{x=1}^{\infty} x q^{x-1} \quad ; \quad \sum_{x=1}^{\infty} x q^{x-1} = \frac{\partial}{\partial q} \sum_{x=0}^{\infty} q^x$

$= pq \frac{\partial}{\partial q} \sum_{x=0}^{\infty} q^x$

$= pq \frac{\partial}{\partial q} [1 + q + q^2 + \dots]$

$\therefore E(x) = pq \frac{\partial}{\partial q}\left(\frac{1}{1-q}\right) = pq \frac{1}{(1-q)^2} = \frac{pq}{p^2} = q/p$

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B - $Var(X) = Ex^2 - (Ex)^2$

$$Ex^2 = \sum_{x=1}^{\infty} x^2 p q^{x-1} = \sum_{x=1}^{\infty} [x(x-1)+x] p q^{x-1}$$

$$= p \sum_{x=1}^{\infty} x(x-1) q^{x-1} + p \sum_{x=1}^{\infty} x q^{x-1}$$

$$\frac{\partial}{\partial q} q^x = x q^{x-1} \; ; \; \frac{\partial^2}{\partial^2 q} q^x = x(x-1) q^{x-2}$$

$$\therefore Ex^2 = p \sum_{x=1}^{\infty} x(x-1) q q^{x-2} + p \sum_{x=1}^{\infty} x q^{x-1}$$

$$= pq \sum_{x=2}^{\infty} x(x-1) q^{x-2} + p \sum_{x=1}^{\infty} x q^{x-1}$$

$$= pq \frac{\partial^2}{\partial^2 q} \sum_{x=0}^{\infty} q^x + pq \frac{\partial}{\partial q} \sum_{x=0}^{\infty} q^x$$

$$= pq \frac{\partial^2}{\partial^2 q} \left(\frac{1}{1-q}\right) + pq \frac{\partial}{\partial q} \left(\frac{1}{1-q}\right)$$

$$= pq \frac{2}{(1-q)^3} + p \frac{q}{(1-q)^2} = \frac{2pq}{p^3} + \frac{pq}{p^2}$$

$$= \frac{2q}{p^2} + \frac{q}{p}$$

C - $\mu_x^t = m_x(t) = E(e^{tx}) = \sum_{x=0}^{\infty} e^{tx} p q^x$

$$= p \sum_{x=0}^{\infty} (e^t q)^x = p[1 + e^t q + (e^t q)^2 + \dots]$$

$$= p \frac{1}{1-e^t q} = p(1-qe^t)^{-1}$$

$$\therefore m_x(t) = p(1-qe^t)^{-1}$$